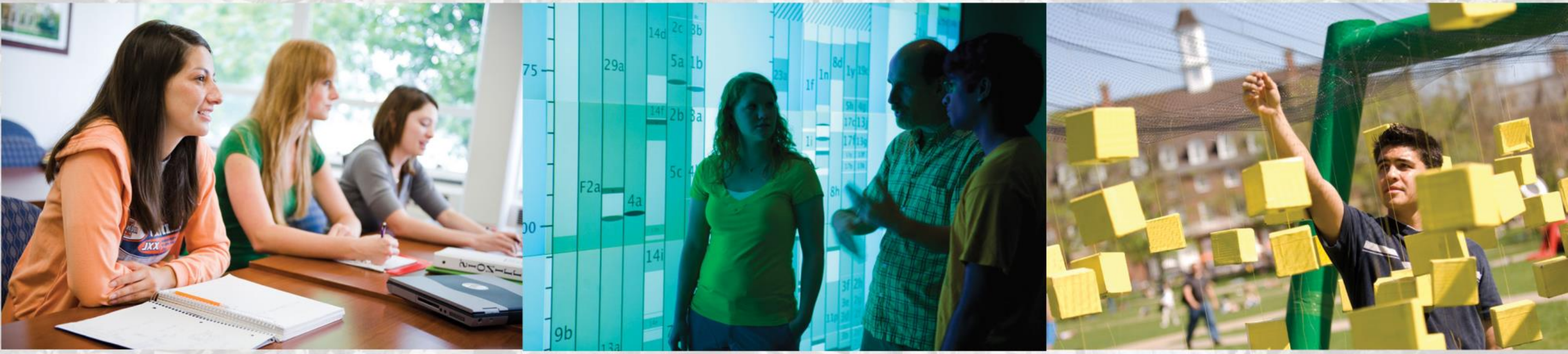
UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

Frequency Domain Analysis of Linear Circuits Using Synchronous Detection

Physics 401, Fall 2019

Eugene V. Colla



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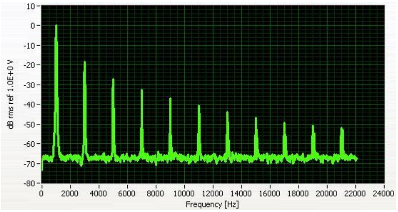
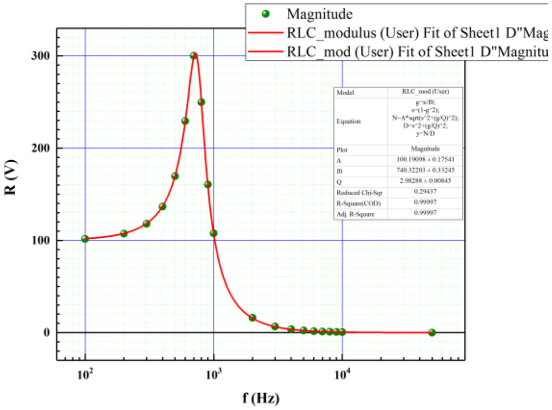
The main issues of this week lab:

1. **Fourier Transform and using FFT in data analysis.**



2. **Lock-in amplifier and frequency domain technique**

3. **Data analysis using OriginPr – nonlinear fitting**



Model	RLC_mod (User)

Equation	g=x/f0;
	s=(1-g^2);
	N=A*sqrt(s^2+(g/Q)^2);
	D=s^2+(g/Q)^2;
	y=N/D

Plot	Magnitude
A	100.19098 ± 0.17541
f0	740.32203 ± 0.33245
Q	2.98288 ± 0.00845
Reduced Chi-Sqr	0.29437
R-Square(COD)	0.99997
Adj. R-Square	0.99997

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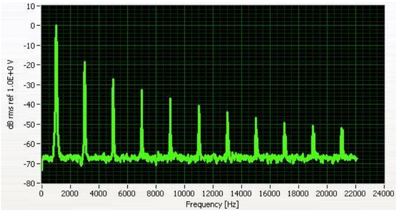

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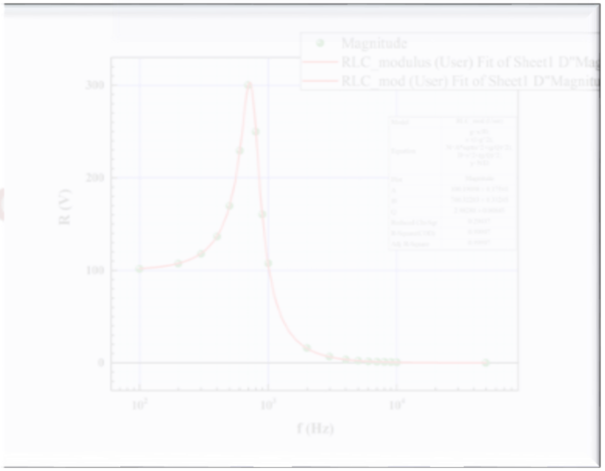
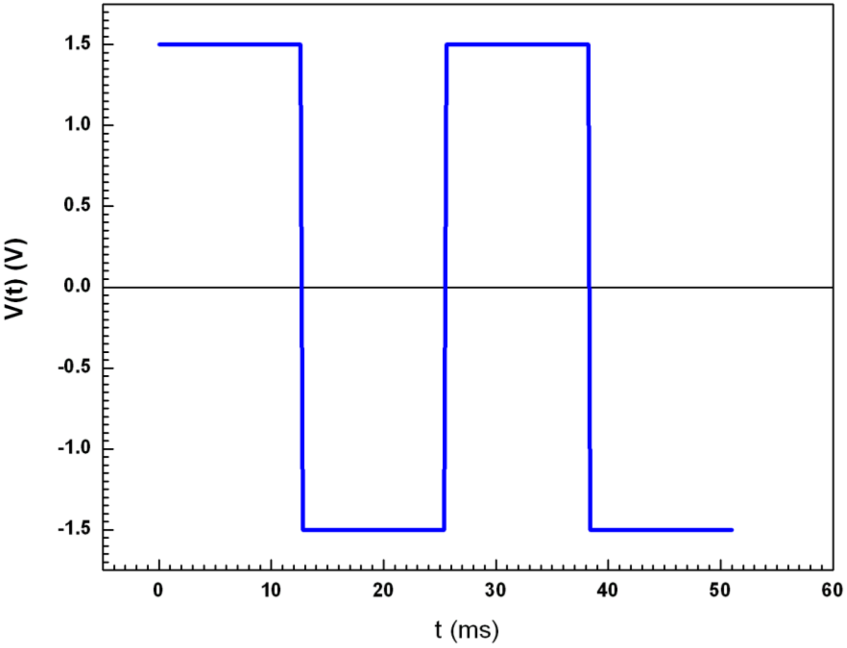


in 1822, Jean Baptiste Fourier developed the theory that shows that any real waveform can be represented by the sum of sinusoidal waves.

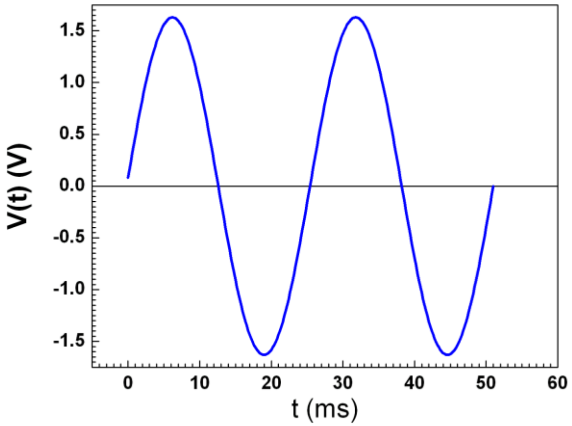
Jean Baptiste Joseph Fourier (1768 – 1830)

Let we try to create the square wave as a sum of sine waves of different frequencies

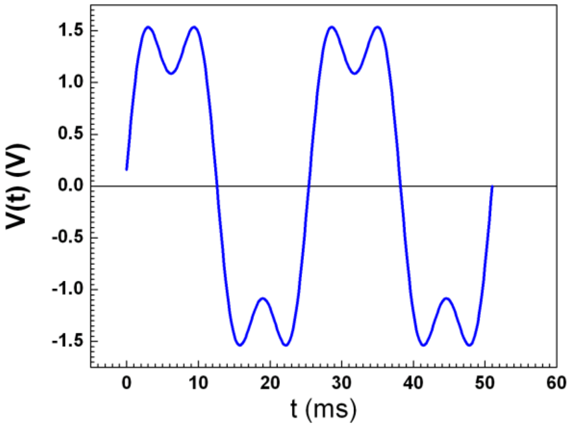
Square wave.
F=40Hz, A=1.5V



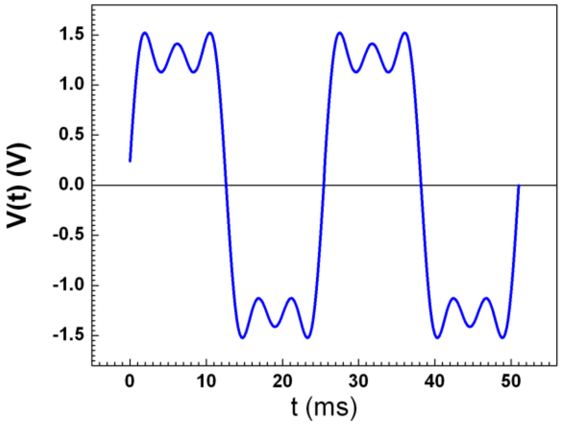
$$A_1\sin(2\pi\omega t)$$



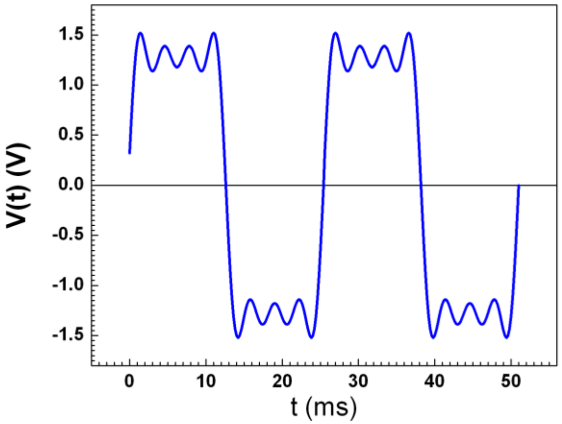
$$A_1\sin(2\pi\omega t) + A_3\sin(2\pi 3\omega t + \varphi_3)$$



$$A_1\sin(2\pi\omega t) + A_3\sin(2\pi 3\omega t + \varphi_3) + A_5\sin(2\pi 5\omega t + \varphi_5)$$



$$A_1\sin(2\pi\omega t) + A_3\sin(2\pi 3\omega t + \varphi_3) + A_5\sin(2\pi 5\omega t + \varphi_5) + A_7\sin(2\pi 7\omega t + \varphi_7)$$



Fourier Transform

The continues *Fourier transformation* of the signal h(t) can be written as:

$$H(f)=\int_{-\infty}^{+\infty} h(t)e^{2\pi jft}dt; \quad j=\sqrt{-1}$$

H(f) represents in frequency domain mode the time domain signal h(t)

Equation for *inverse Fourier transform* gives the correspondence of the infinite continues frequency spectra to the corresponding time domain signal.

$$h(t)=\int_{-\infty}^{+\infty} H(f)e^{-2\pi jft}df$$

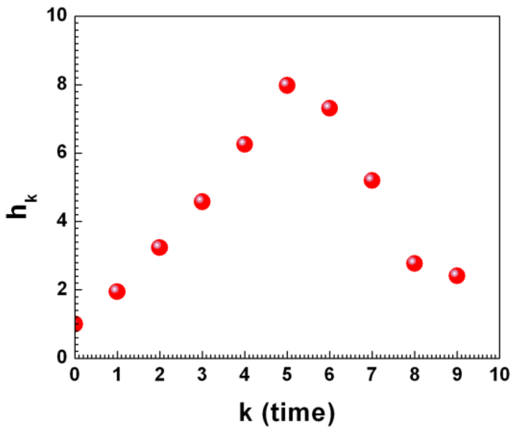
In real life we working with discrete representation of the time domain signal recorded during a finite time.

Discrete Fourier Transform

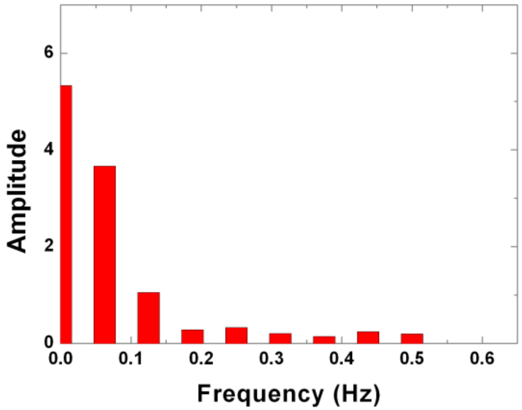
It comes out that in practice more useful is the representation the frequency domain pattern of the time domain signal h_k as sum of the frequency harmonic calculated as:

$$H_n = H(f_n) = \frac{1}{N}\sum_{k=0}^{N-1} h_k e^{2\pi kn/N}$$

Δ is the sampling interval, N – number of collected points



Time domain



Frequency domain

Discrete Fourier Transform

For periodic signals with period T_0:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_0}\right)$$

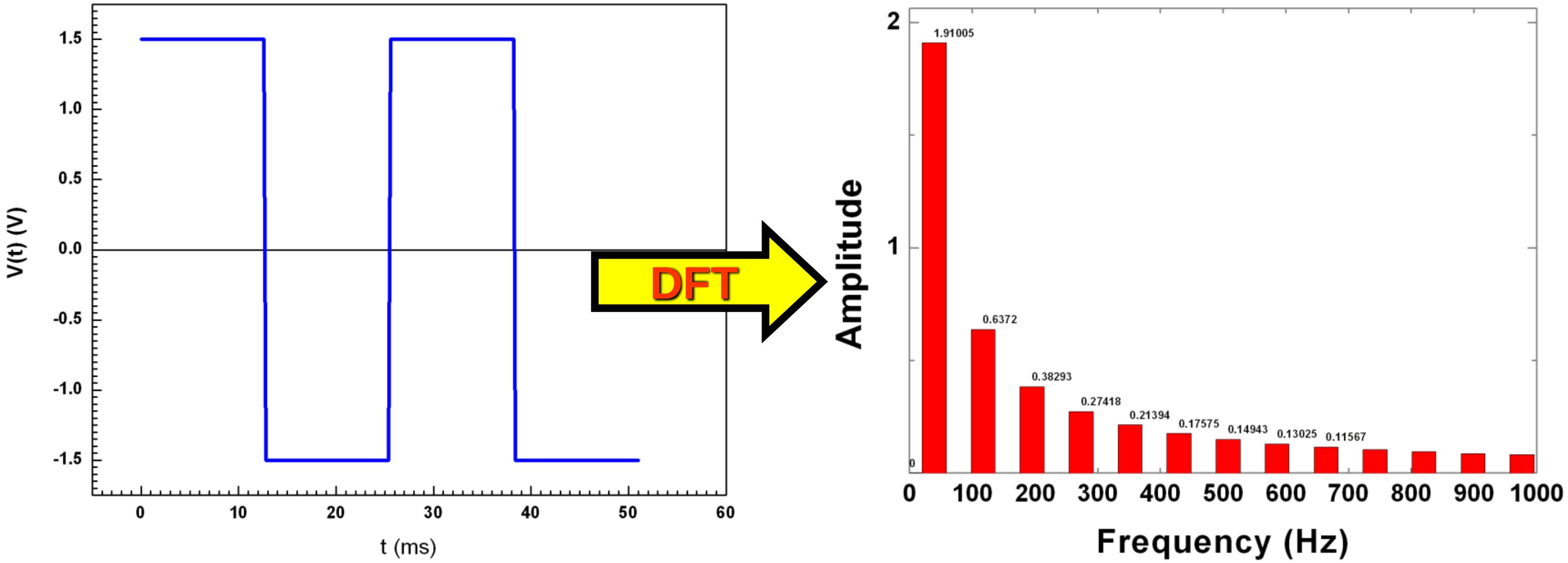
$$a_n = \frac{2}{T_0}\int_0^{T_0} F(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt; \qquad b_n = \frac{2}{T_0}\int_0^{T_0} F(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt;$$

$$a_0 = \frac{2}{T_0}\int_0^{T_0} F(t)dt;$$

Discrete Fourier Transform

Now how I found the amplitudes of the harmonics to compose the square wave signal from sine waves of different frequencies.

Time domain signal



Decomposition the signal into the sine wave harmonics. The only modulus's of the harmonics amplitudes are presented in this picture.

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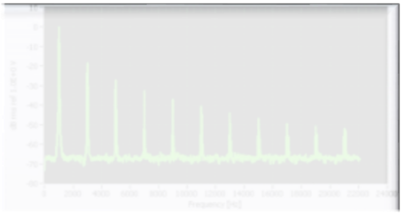
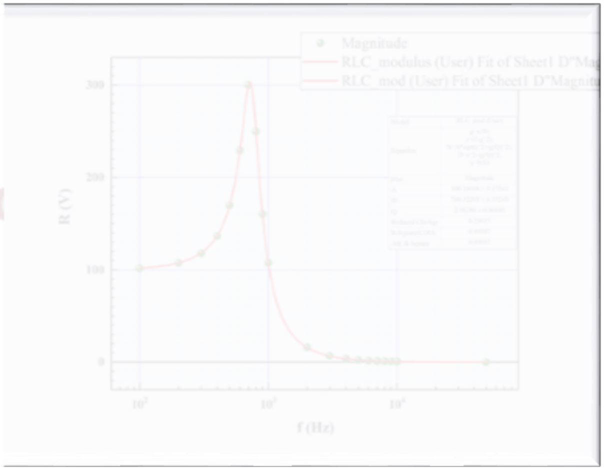


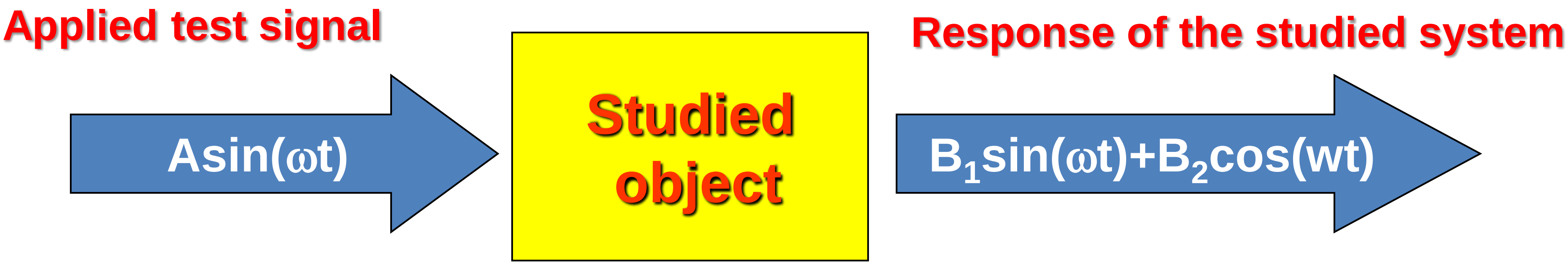
2. **Lock-in amplifier and frequency domain technique**

3. Data analysis using OriginPr
– nonlinear fitting



Frequency Domain Spectroscopy

(linear system)



Applied test signal

$A\sin(\omega t)$

Studied object

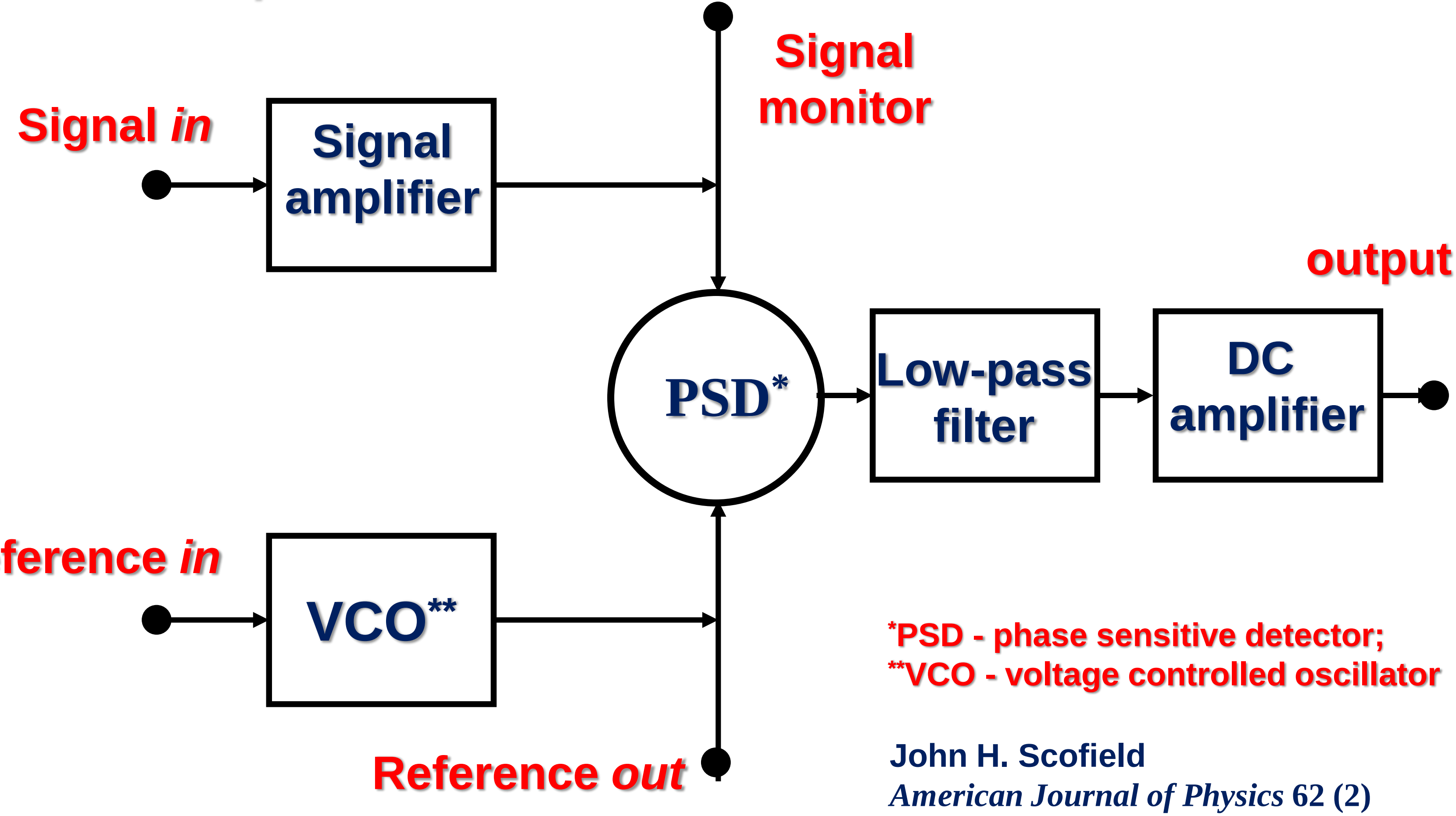
Response of the studied system

$B_1\sin(\omega t)+B_2\cos(wt)$

We applying the sine wave signal to the tested object and measuring the response. Varying the frequency we can study the frequency properties of the system.

Lock-in amplifier

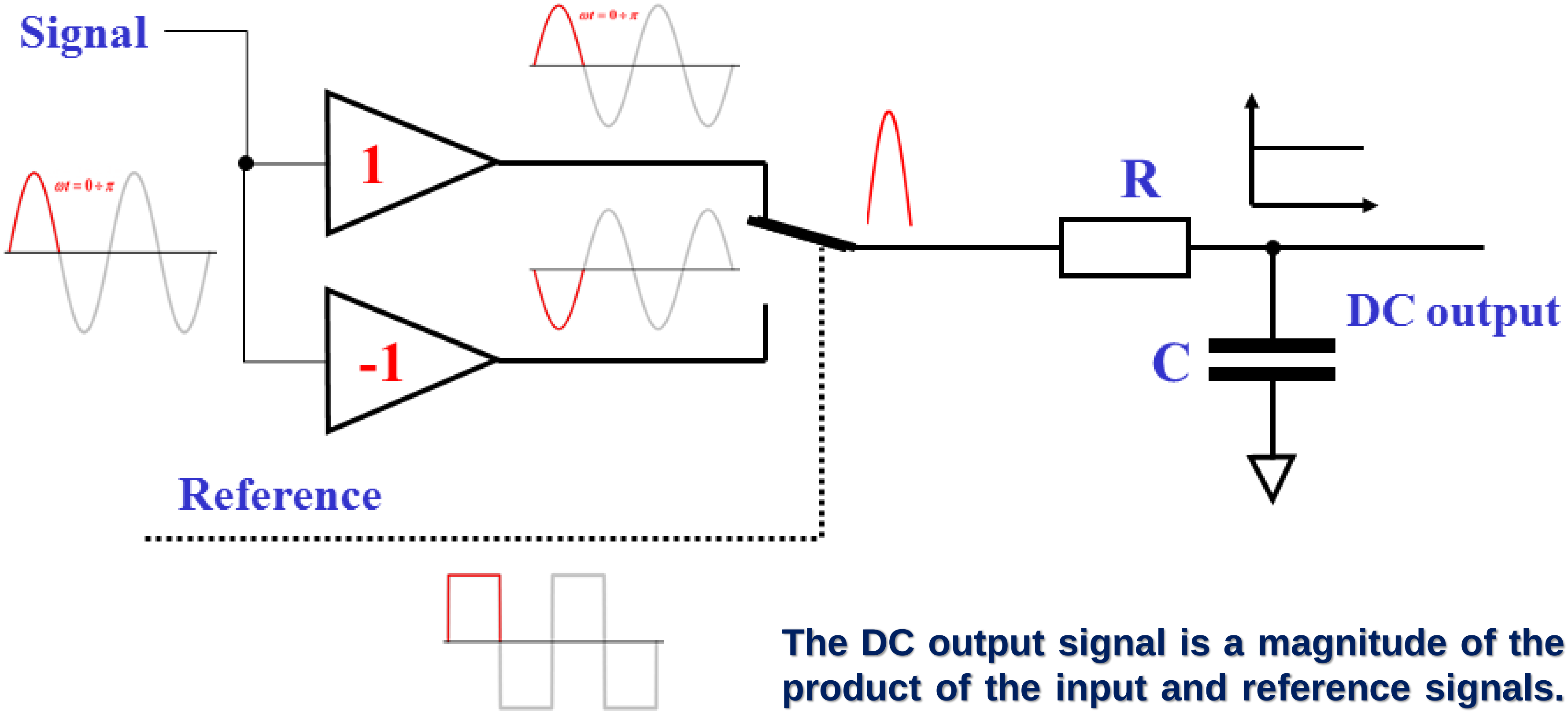
Now about the most powerful tool which can be used in frequency domain technique.



*PSD - phase sensitive detector;
**VCO - voltage controlled oscillator

John H. Scofield
American Journal of Physics 62 (2)
129-133 (Feb. 1994).

Lock-in amplifier. How it works.

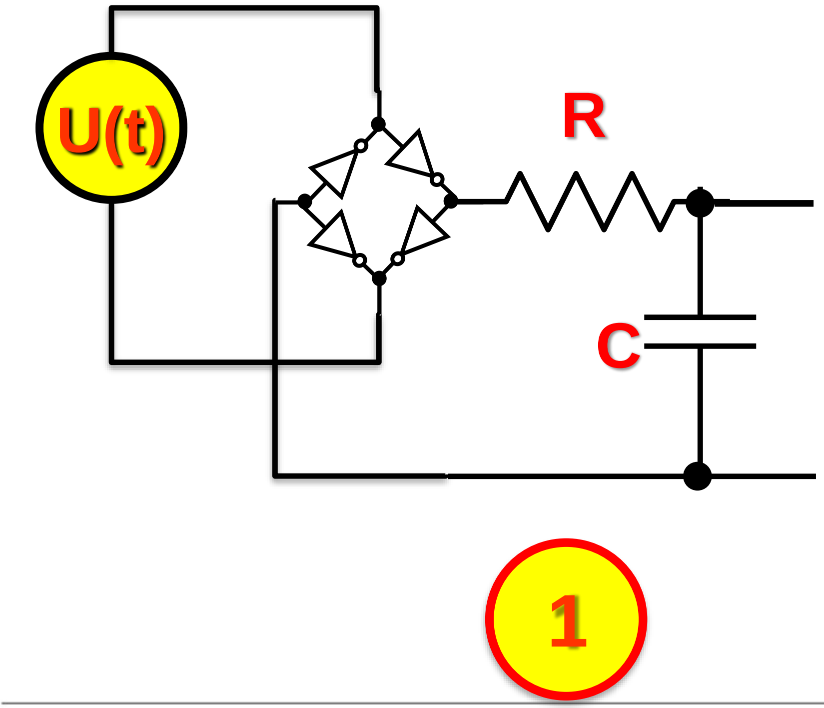


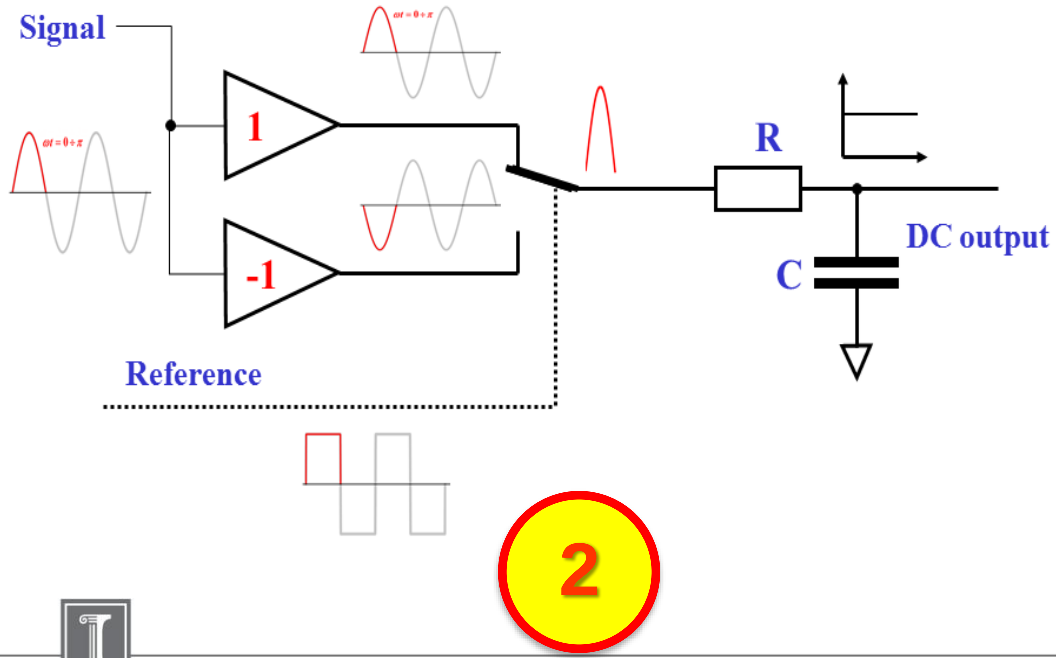
The DC output signal is a magnitude of the product of the input and reference signals. AC components of output signal are filtered out by the low-pass filter with time constant τ (her τ=RC)

Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

DMM, lock-in etc.

Results as DC voltage corresponding U_{AMP}, U_{RMS} …

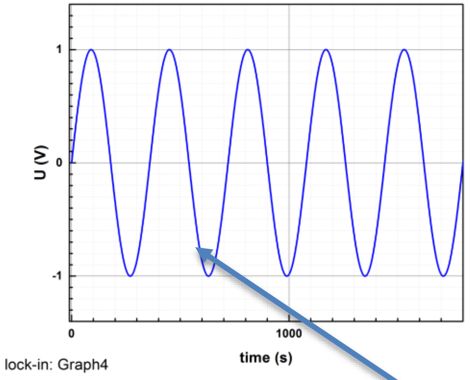


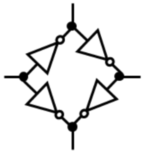


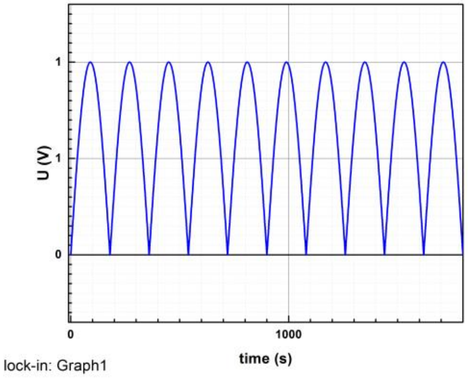
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

1

Clean sine wave – no “noise”







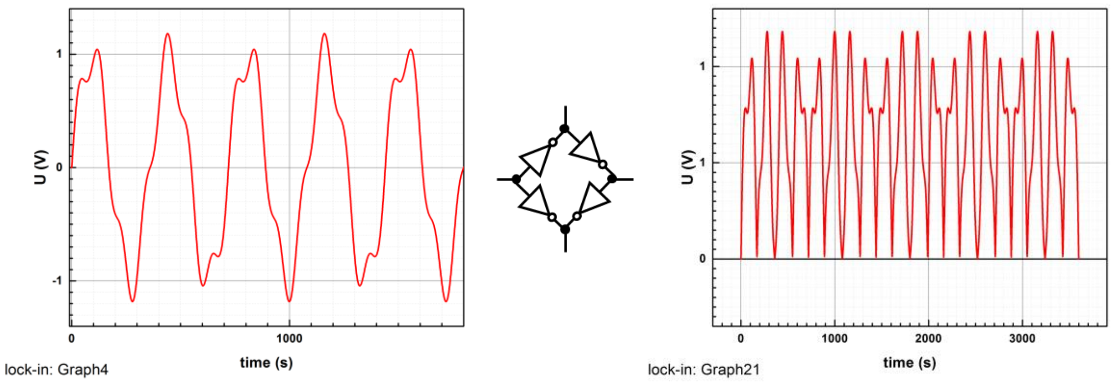
U_{DC}=0.63643V

We need to measure the amplitude/rms value of the sine wave

Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

1

"Noisy" sine wave



U_{DC}=0.64208V

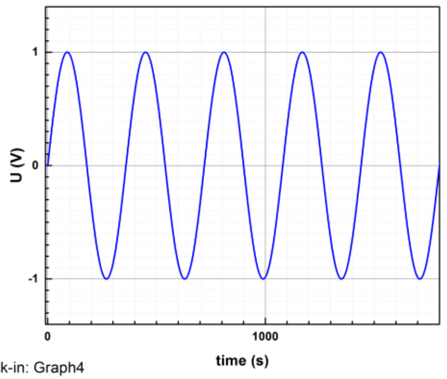
compare to

U_{DC}=0.63643V

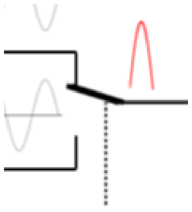
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

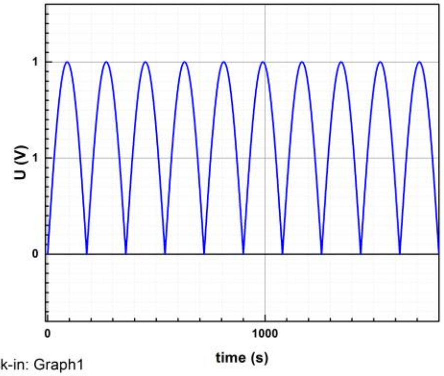
2

Clear sine wave – no “noise”



lock-in: Graph4



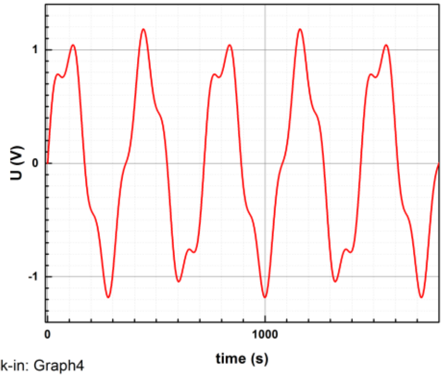


lock-in: Graph1

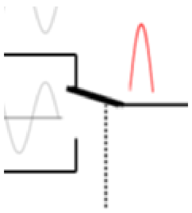
U_{DC}=0.63643V

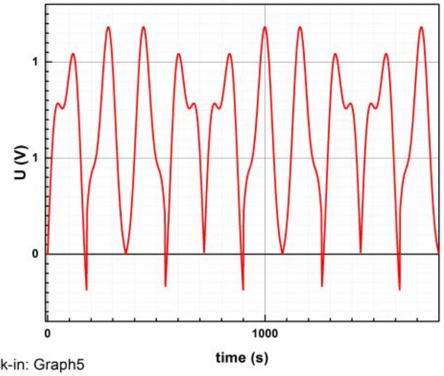
=

“Noisy” sine wave



lock-in: Graph4

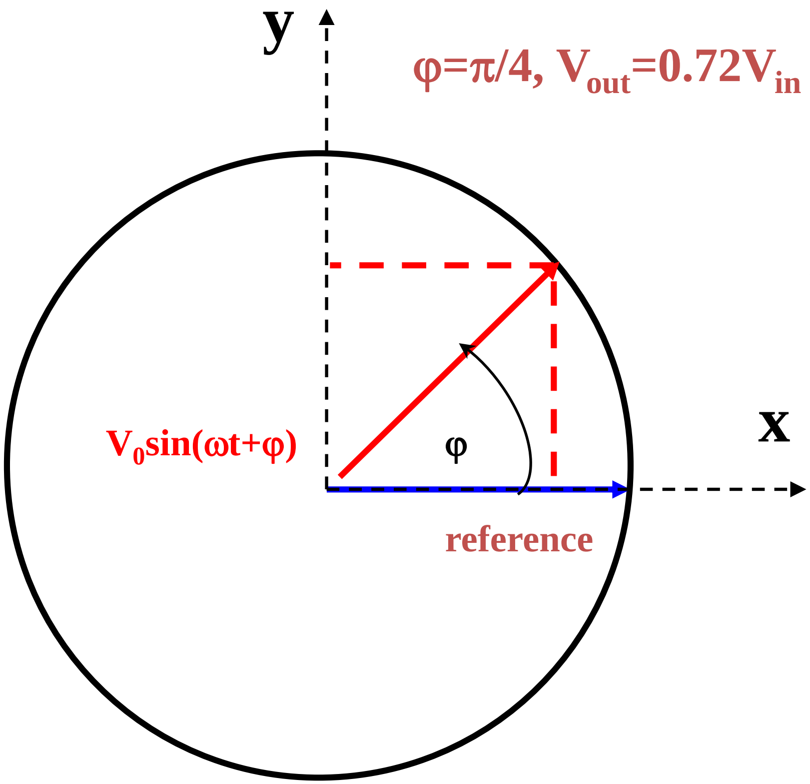


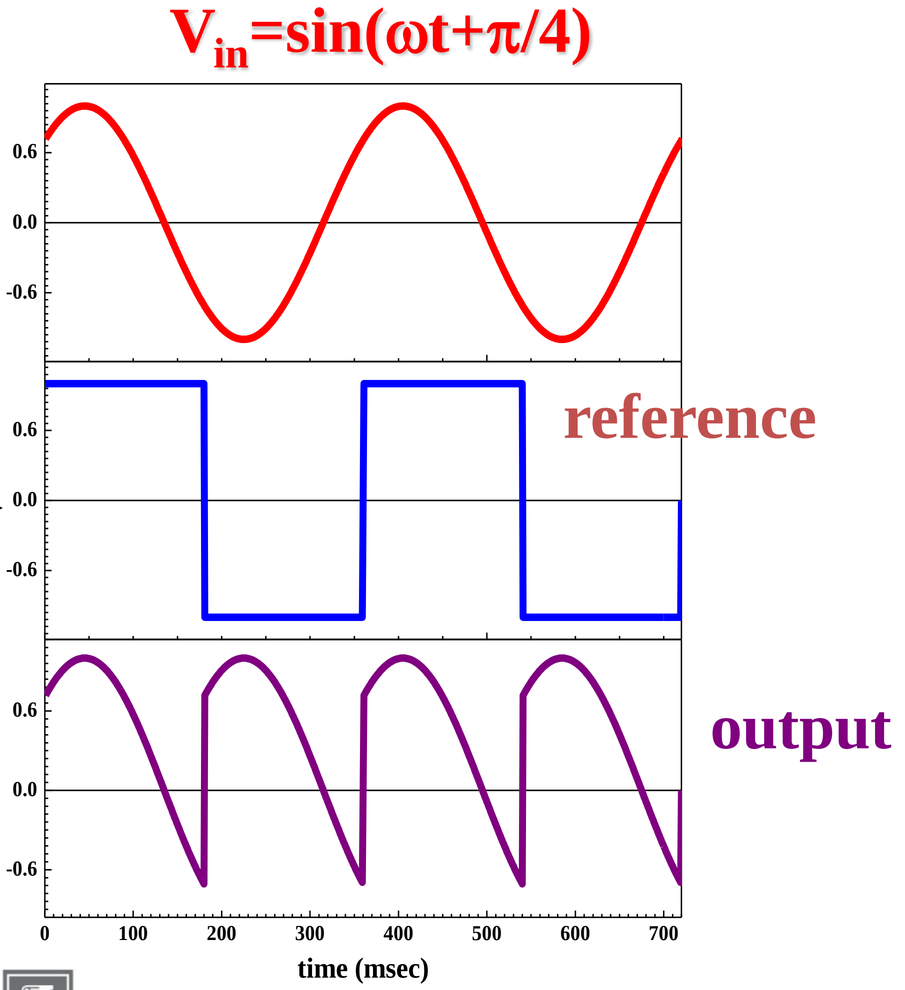


lock-in: Graph5

U_{DC}=0.63643V

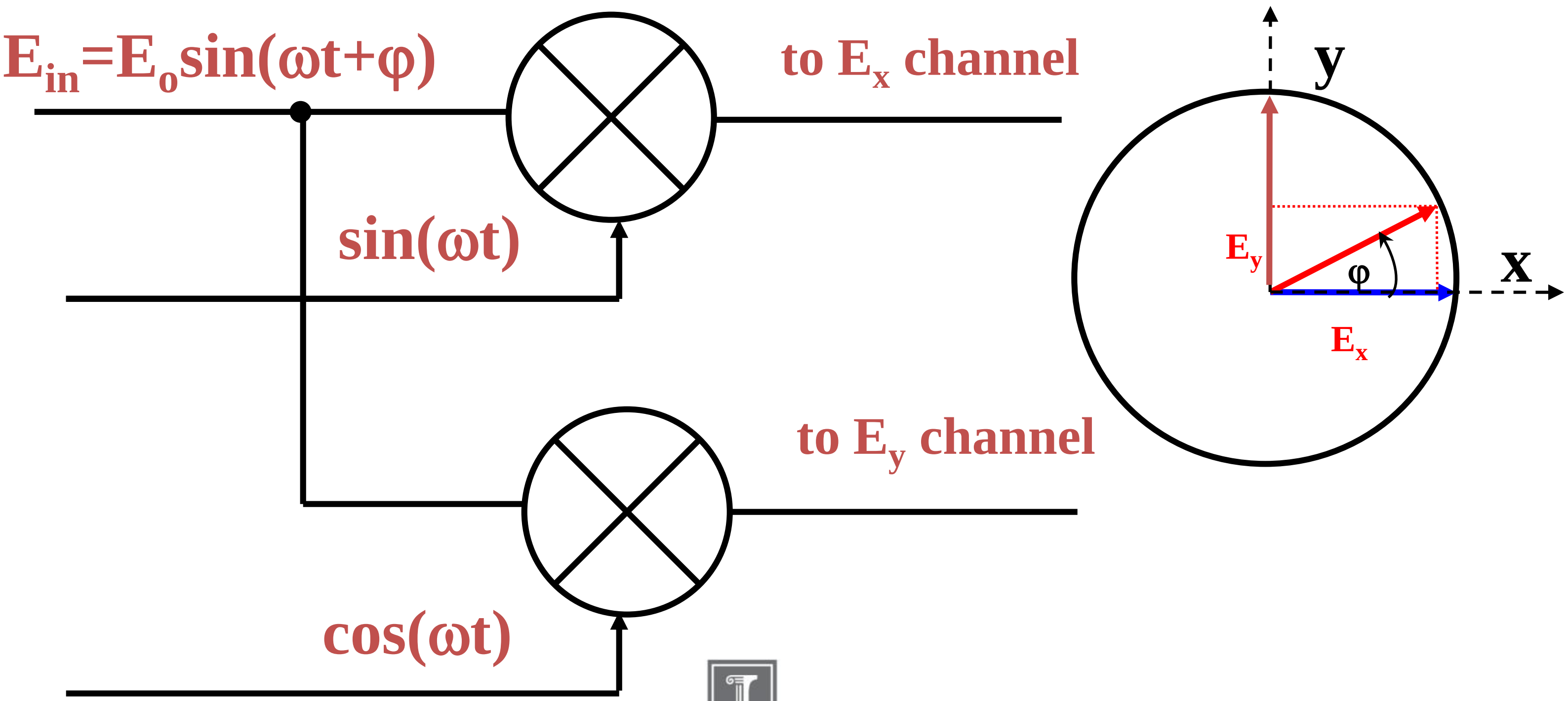
Lock-in Amplifier. Phase shift.



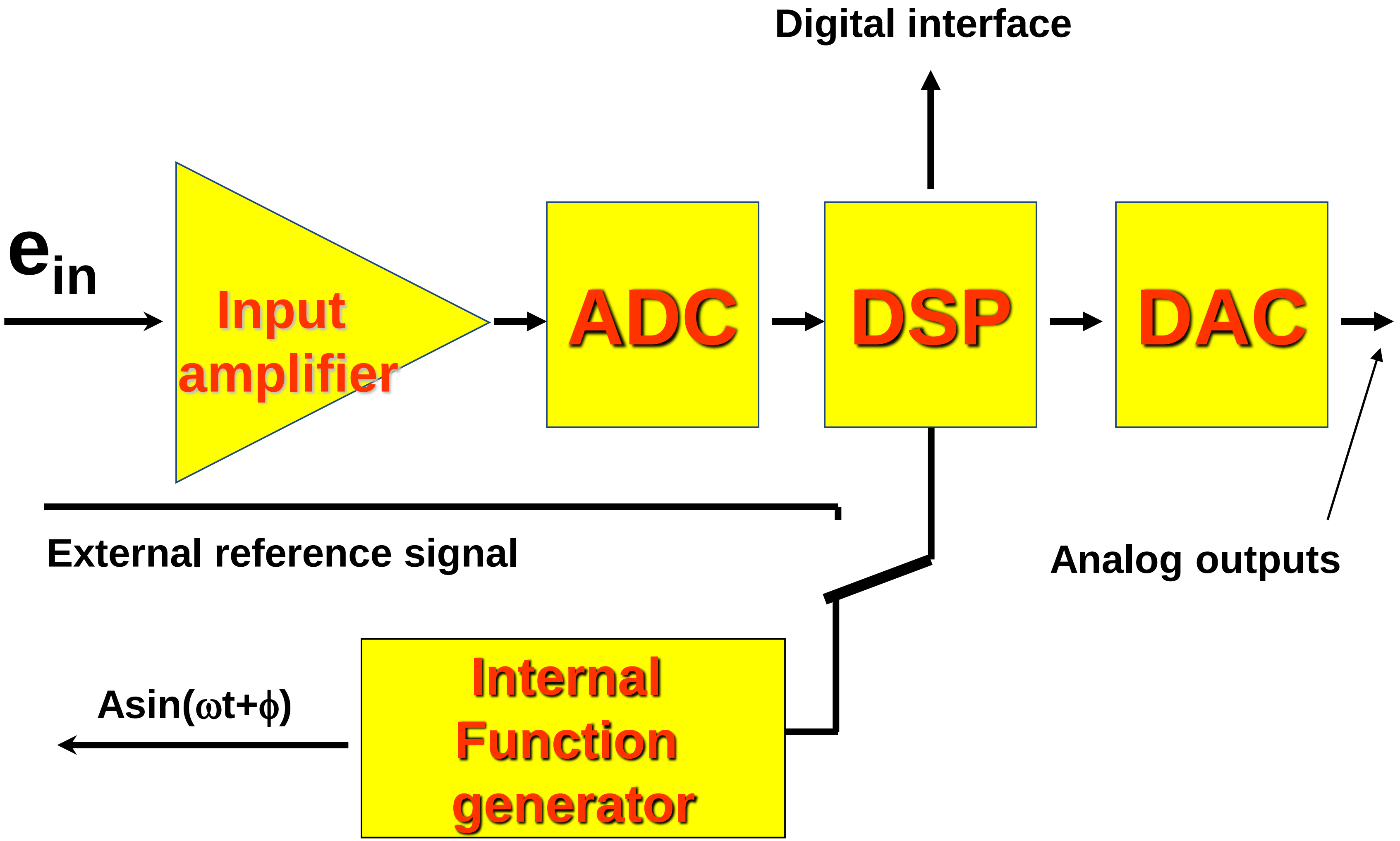


Lock-in Amplifier. Two Channels Demodulation.

In many scientific applications it is a great advantage to measure both components (E_x, E_y) of the input signal. We can use two lock-ins to do this or we can measure these value in two steps providing the phase shift of reference signal 0 and π/2. Much better solution is to use the lock-in amplifier equipped by two demodulators.



Digital Lock-in Amplifier



Digital interface

e_{in}

Input amplifier

ADC

DSP

DAC

External reference signal

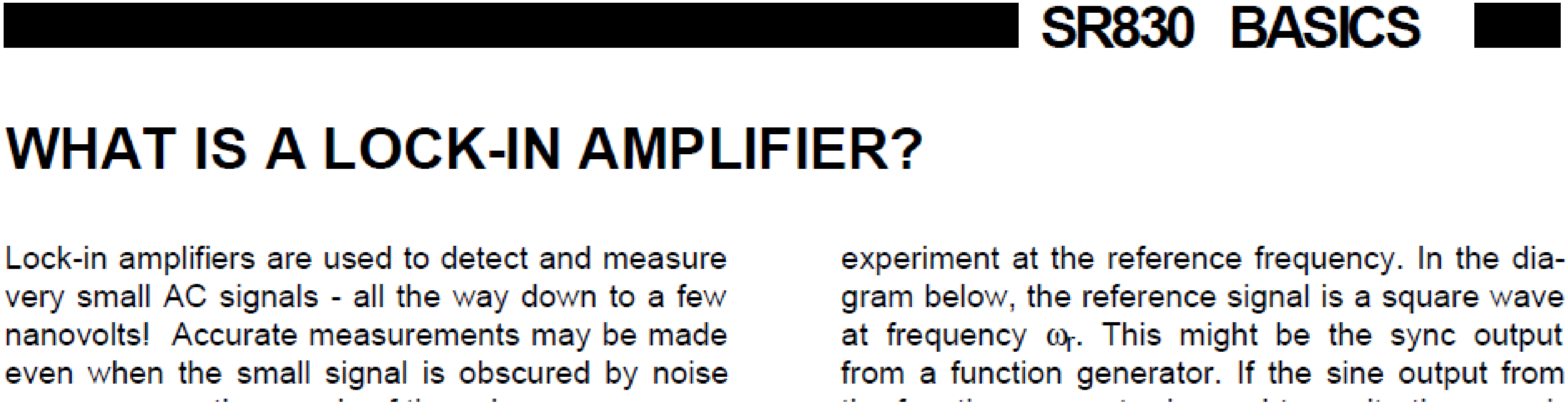
Analog outputs

Asin(ωt+ϕ)

Internal Function generator

SR830. Digital Lock-in Amplifier

In SR830 manual you can find the chapter dedicated to general description of the lock-in amplifier idea



SR830 BASICS

WHAT IS A LOCK-IN AMPLIFIER?

Lock-in amplifiers are used to detect and measure very small AC signals - all the way down to a few nanovolts! Accurate measurements may be made even when the small signal is obscured by noise

experiment at the reference frequency. In the diagram below, the reference signal is a square wave at frequency ω_r. This might be the sync output from a function generator. If the sine output from

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Digital Lock-in amplifier. SR830

Time constant
And output filter

sensitivity

Channel#1

Channel#2

Auto functions

Inputs

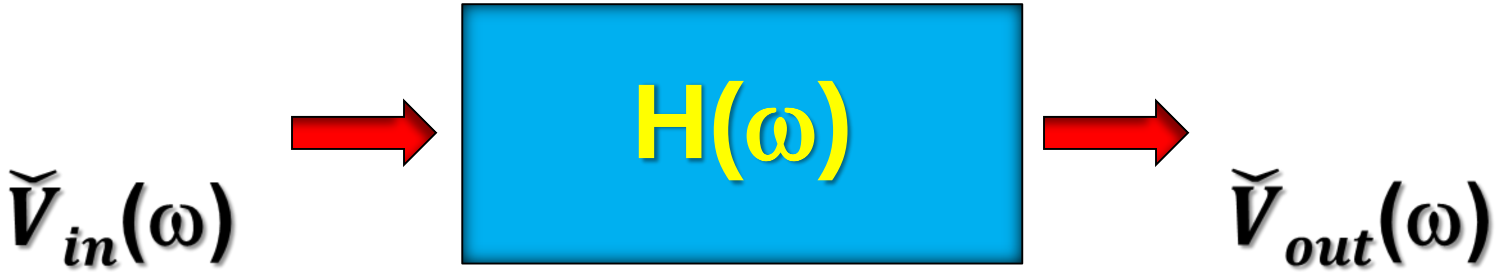
Notch filter settings

Analog outputs

Interface settings

Function generator

Experiments. Main idea. Investigating the frequency response of circuit.



$\breve{V}_{in}(\omega)$ → H(ω) → $\breve{V}_{out}(\omega)$

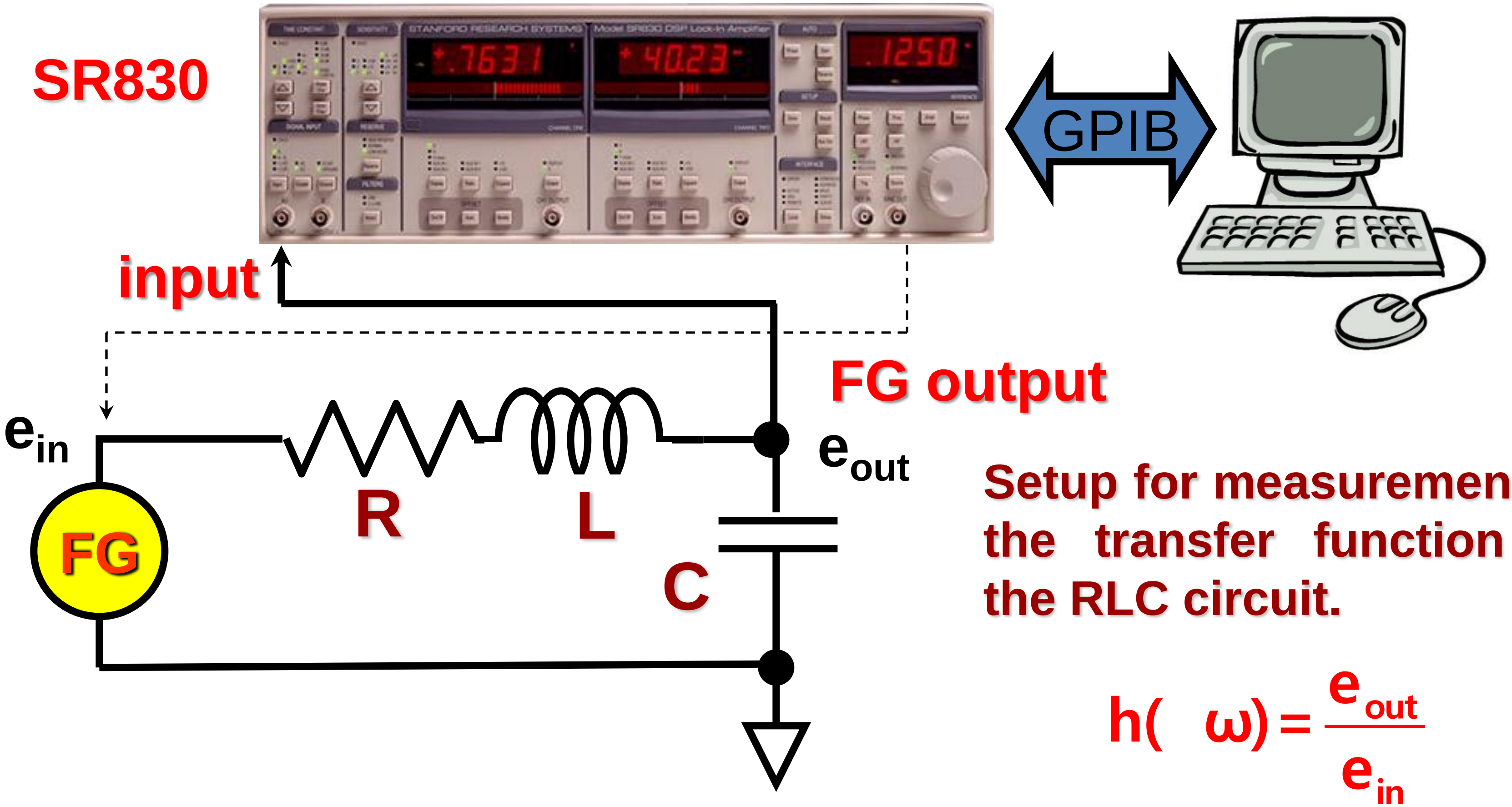
Frequency domain representation of the system

Response function → $$\breve{H}(\omega) = \frac{\breve{V}_{out}(\omega)}{\breve{V}_{in}(\omega)}$$

and $$\breve{V}_{out}(\omega) = \breve{H}(\omega) \bullet \breve{V}_{in}(\omega)$$

Linear systems are those that can be modeled by linear differential equations.

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit



Setup for measurement of the transfer function of the RLC circuit.

$$h(\omega) = \frac{e_{out}}{e_{in}}$$

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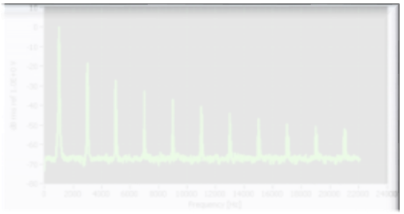

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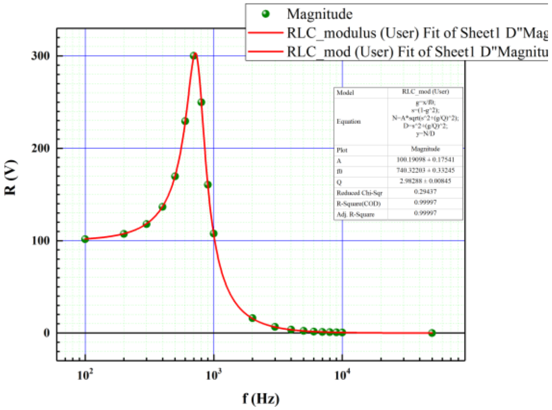

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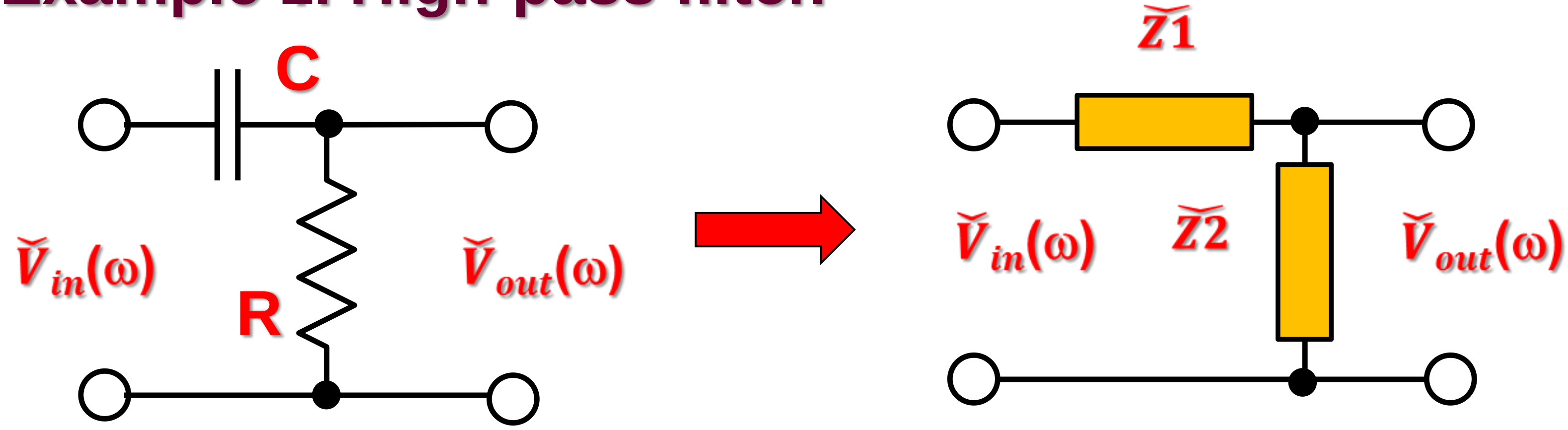
Reduced Chi-Sqr	0.29437
R-Square(COD)	0.99997
Adj. R-Square	0.99997

Experiments. Main Idea. Calculation of the Response Function in Frequency Domain Mode.

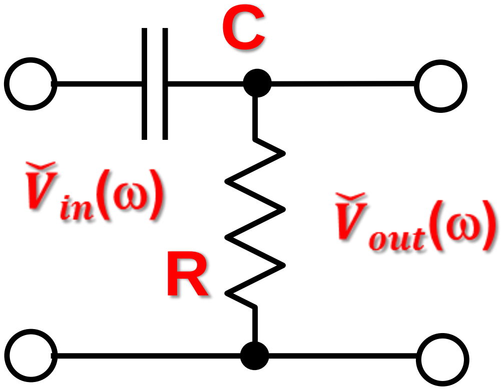
Example 1. High-pass filter.



Applying the Kirchhoff Law to this simple network

$$\tilde{V}_{out}(\omega) = \tilde{H}(\omega) * \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega) \frac{\tilde{Z}2(\omega)}{\tilde{Z}1(\omega) + \tilde{Z}2(\omega)}$$

Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter



Ideal case

$$\tilde{Z}_R = R$$

$$\tilde{Z}_L = j\omega L$$

$$\tilde{Z}_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}$$

More realistic

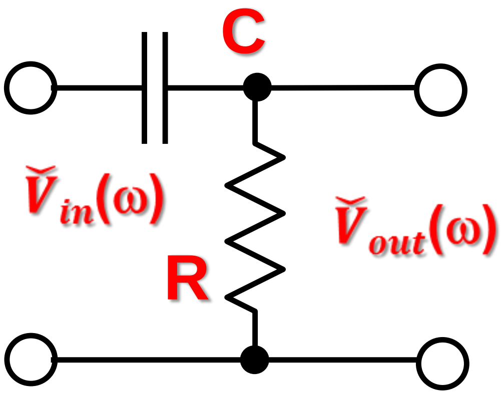
$$\tilde{Z}_R = R + \ldots$$

$$\tilde{Z}_L = j\omega L + R_L$$

$$\tilde{Z}_C = \frac{1}{j\omega C} = \frac{1}{j\omega C + R_C^{-1}}$$

$$\tilde{V}_{out}(\omega) = \tilde{H}(\omega) * \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega)\frac{\tilde{Z}2(\omega)}{\tilde{Z}1(\omega) + \tilde{Z}2(\omega)}$$

Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter



τ – time constant of the filter

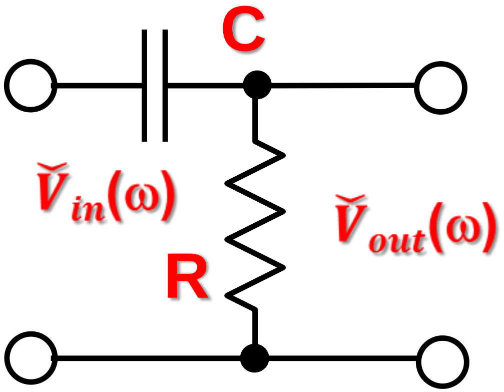
ω_C - cutoff frequency

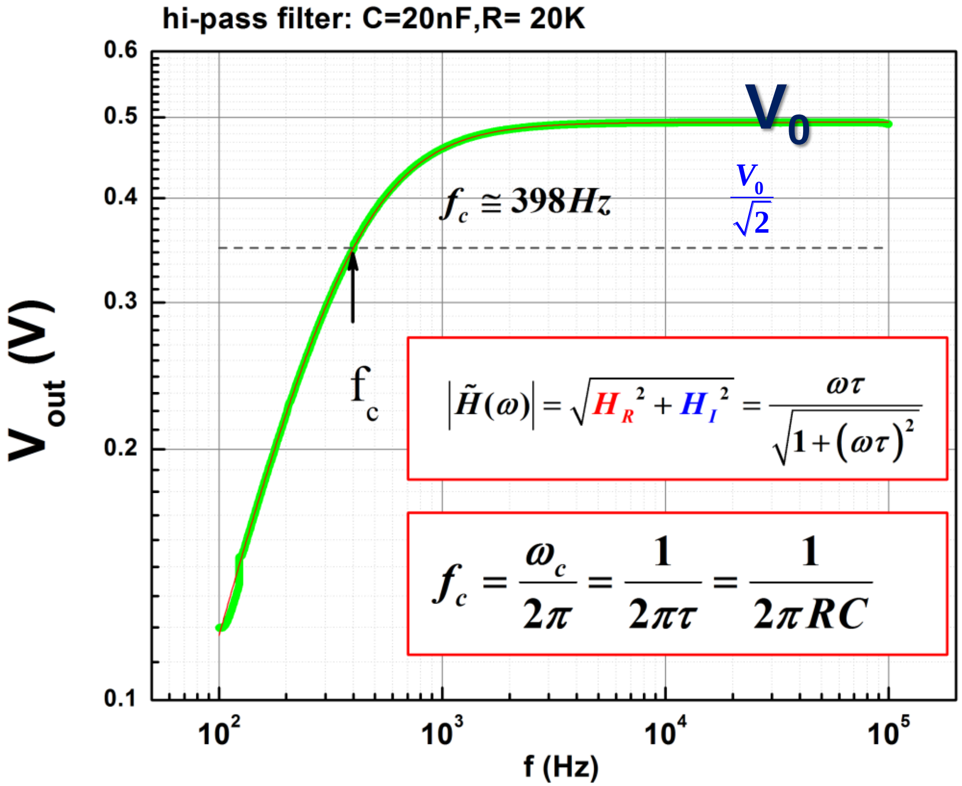
$$\tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{R}{R + \dfrac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega\tau}{1 + j\omega\tau} = \frac{\omega\tau}{\left(1 + \omega^2\tau^2\right)}\left(\omega\tau + j\right);$$

where $\tau = RC = \omega_c^{-1}$;

$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R^{\,2} + H_I^{\,2}} = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = \arctan\left(\frac{1}{\omega\tau}\right)$$

Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter

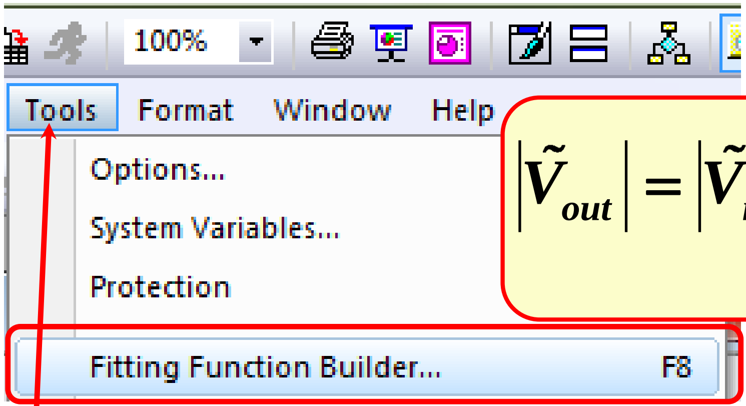




$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R^{\,2} + H_I^{\,2}} = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$$

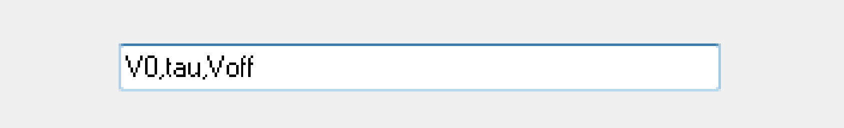
$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\tau} = \frac{1}{2\pi RC}$$

High-pass Filter. Fitting.

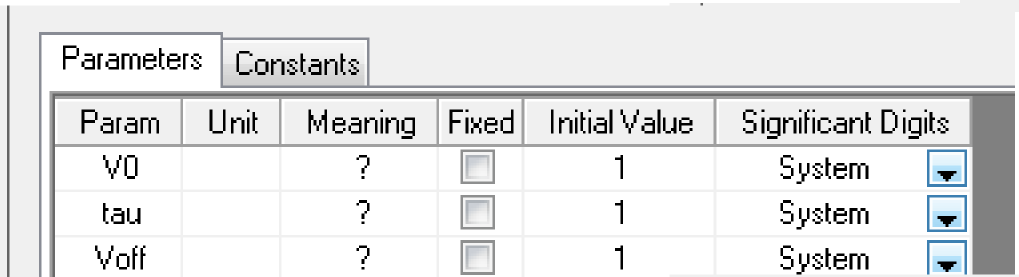


$$\left|\tilde{V}_{out}\right| = \left|\tilde{V}_{in}\right| * \left|\tilde{H}(\omega)\right| = V_0 * \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}; \quad \tau = RC$$

Fitting parameters: V_0, τ, Voff

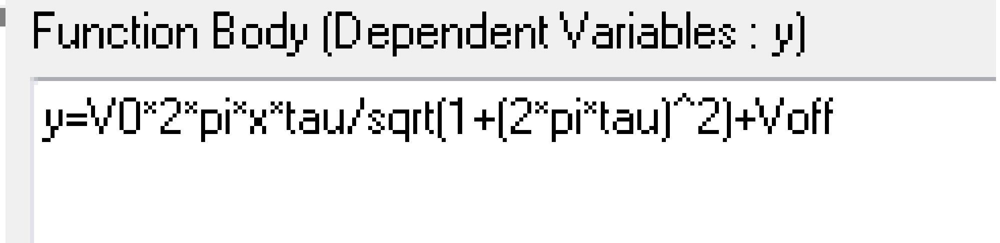


V0,tau,Voff



Param	Unit	Meaning	Fixed	Initial Value	Significant Digits
V0		?	☐	1	System
tau		?	☐	1	System
Voff		?	☐	1	System

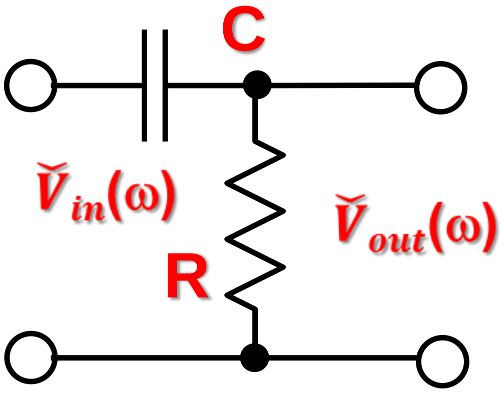
Fitting function →

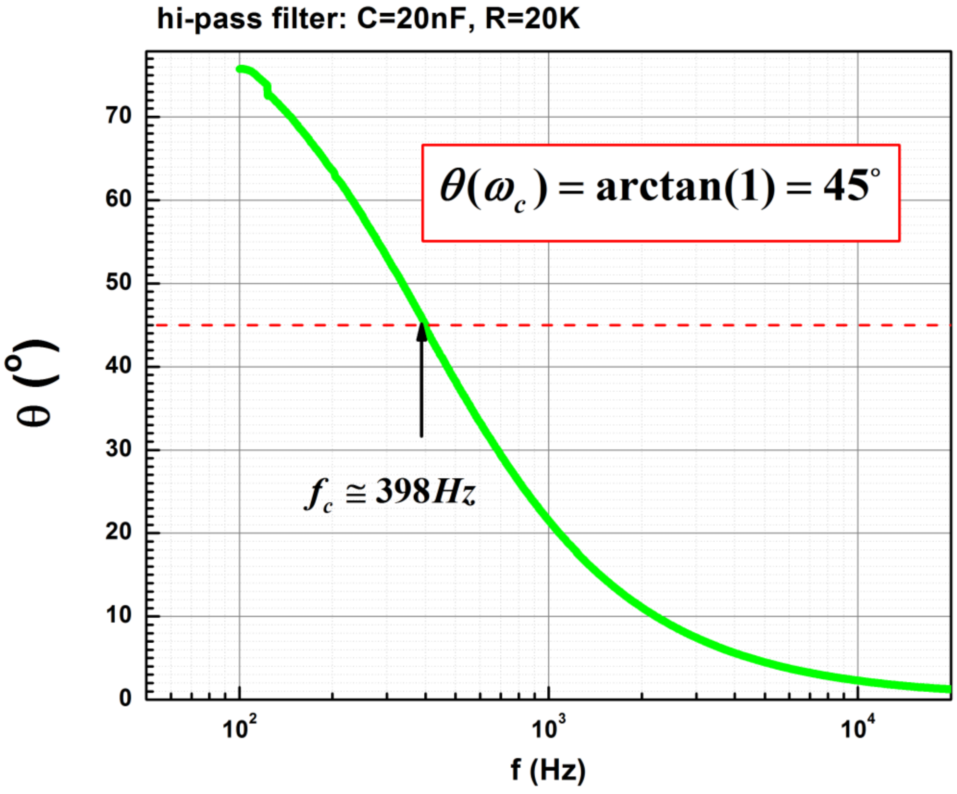


Function Body (Dependent Variables : y)

y=V0*2*pi*x*tau/sqrt(1+(2*pi*tau)^2)+Voff

Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter.

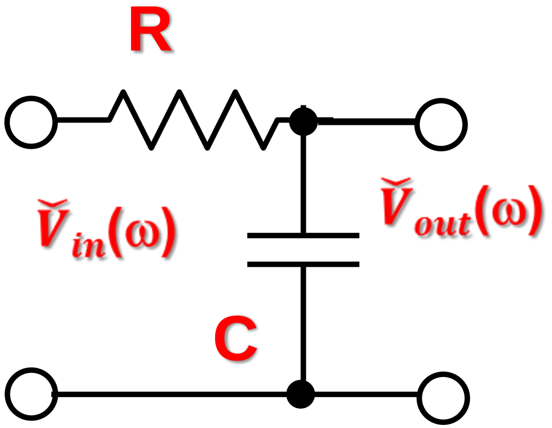


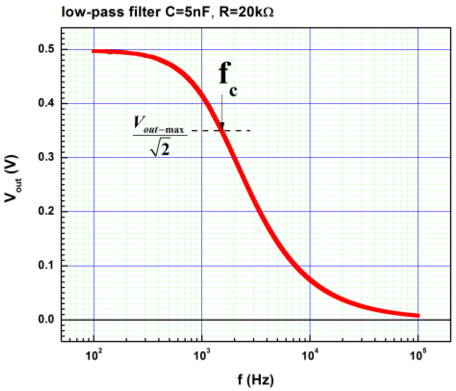


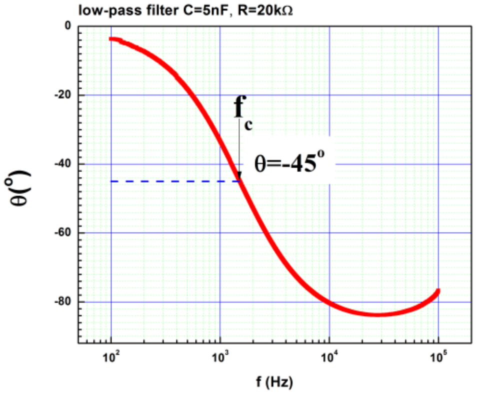
$\theta(\omega_c) = \arctan(1) = 45^\circ$

$f_c \cong 398 Hz$

Experiments. Calculation of the Response Function in Frequency Domain Mode. Low-pass Filter





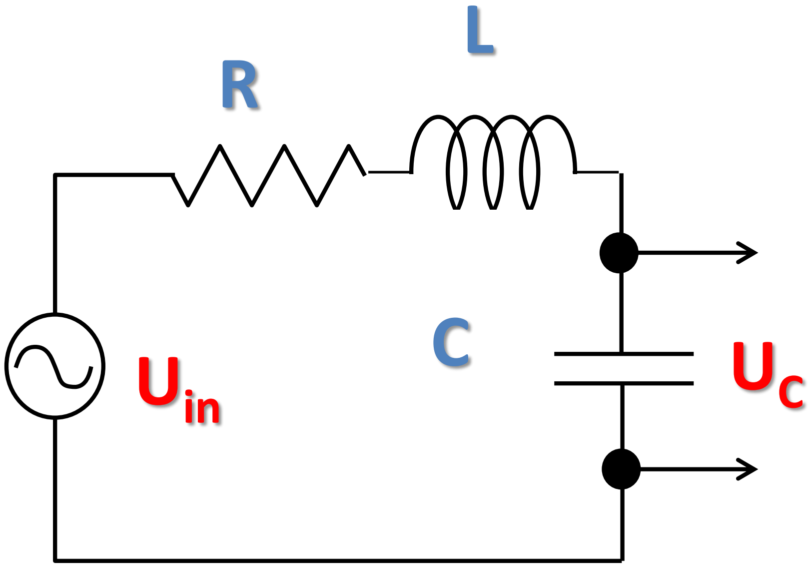


$$\tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega\tau} = \frac{(1 - j\omega\tau)}{(1 + \omega^2\tau^2)};$$

where $\tau = RC = \omega_c^{-1}$;

$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R^{\,2} + H_I^{\,2}} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = -\arctan(\omega\tau)$$

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit .



$$U_C = U_{in} \bullet \frac{Z_C}{Z_C + Z_L + R} =$$

$$= \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L + R}$$

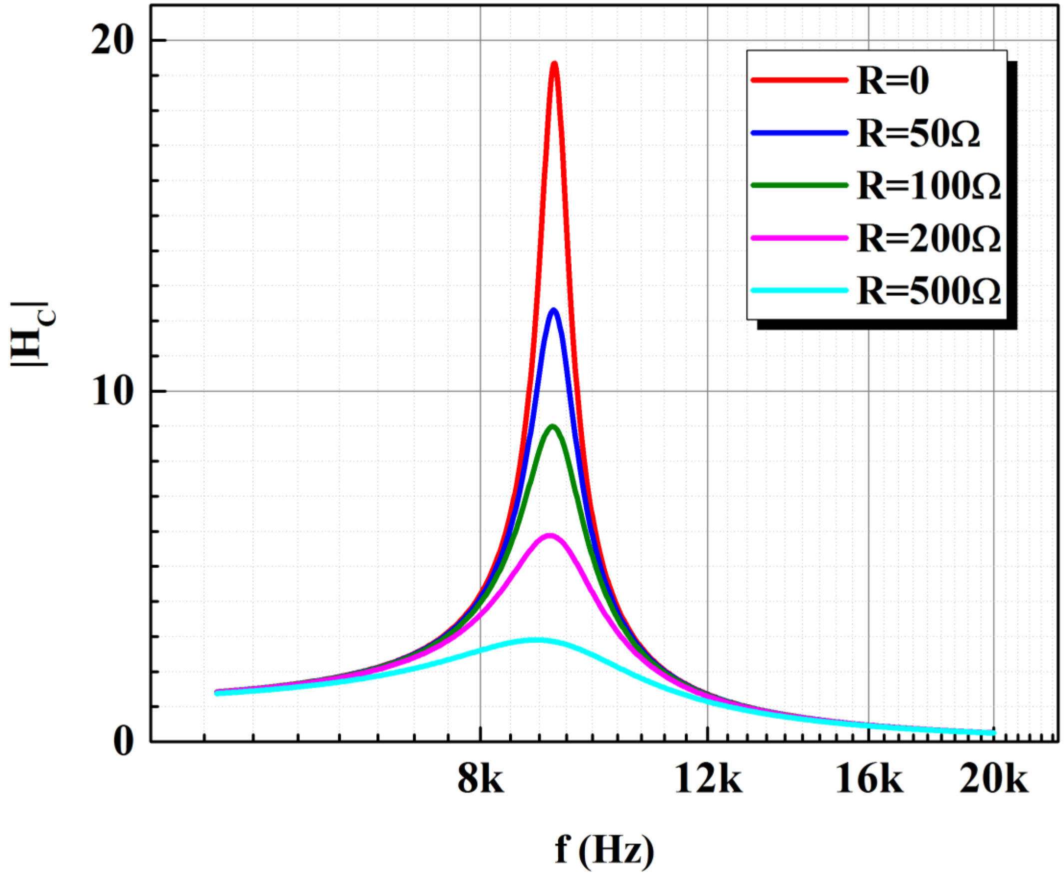
Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit .

$$H=\frac{U_C}{U_{in}}=\frac{1}{(1-\omega^2LC)+j\omega CR}=\frac{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)-j\omega CR}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\omega^2C^2R^2}\times;$$

$$\omega_0=\frac{1}{\sqrt{LC}};\nu\equiv\frac{\omega}{\omega_0};Q=\frac{1}{R}\sqrt{\frac{L}{C}};$$

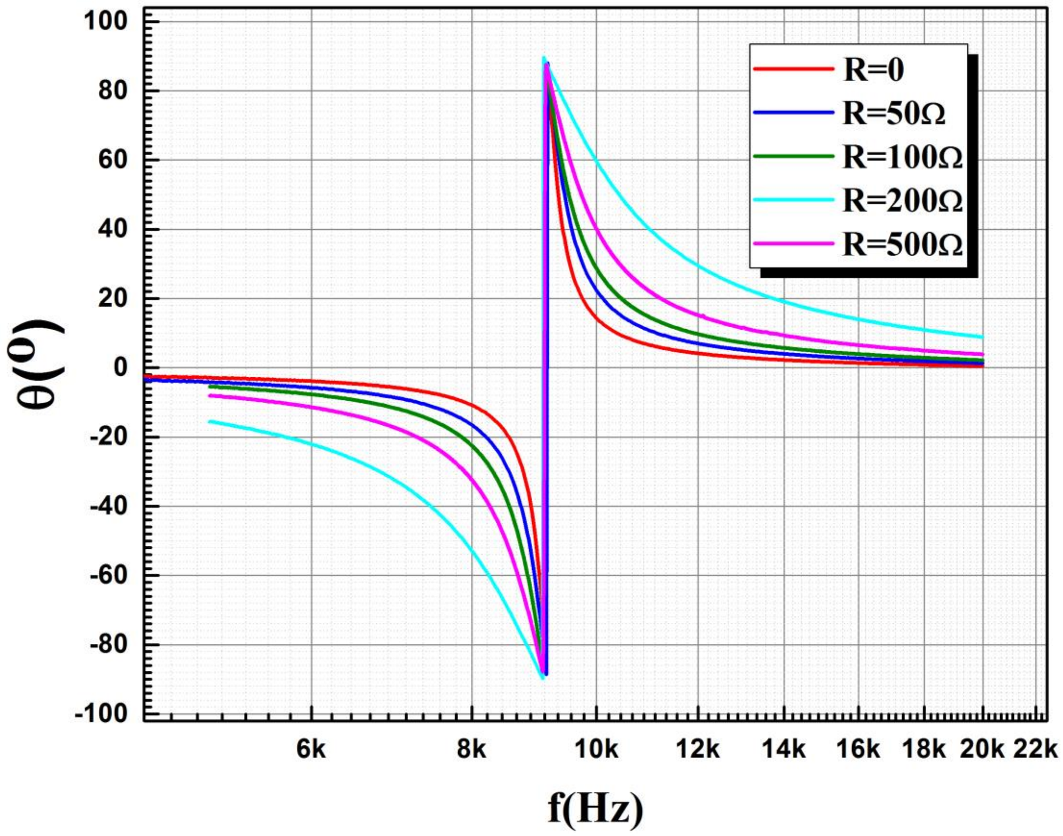
$$H=\frac{(1-\nu^2)-j\frac{\nu}{Q}}{(1-\nu^2)^2+\frac{\nu^2}{Q^2}};\theta=-\tan^{-1}\left(\frac{\nu}{Q\left(1-\nu^2\right)}\right)$$

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit .



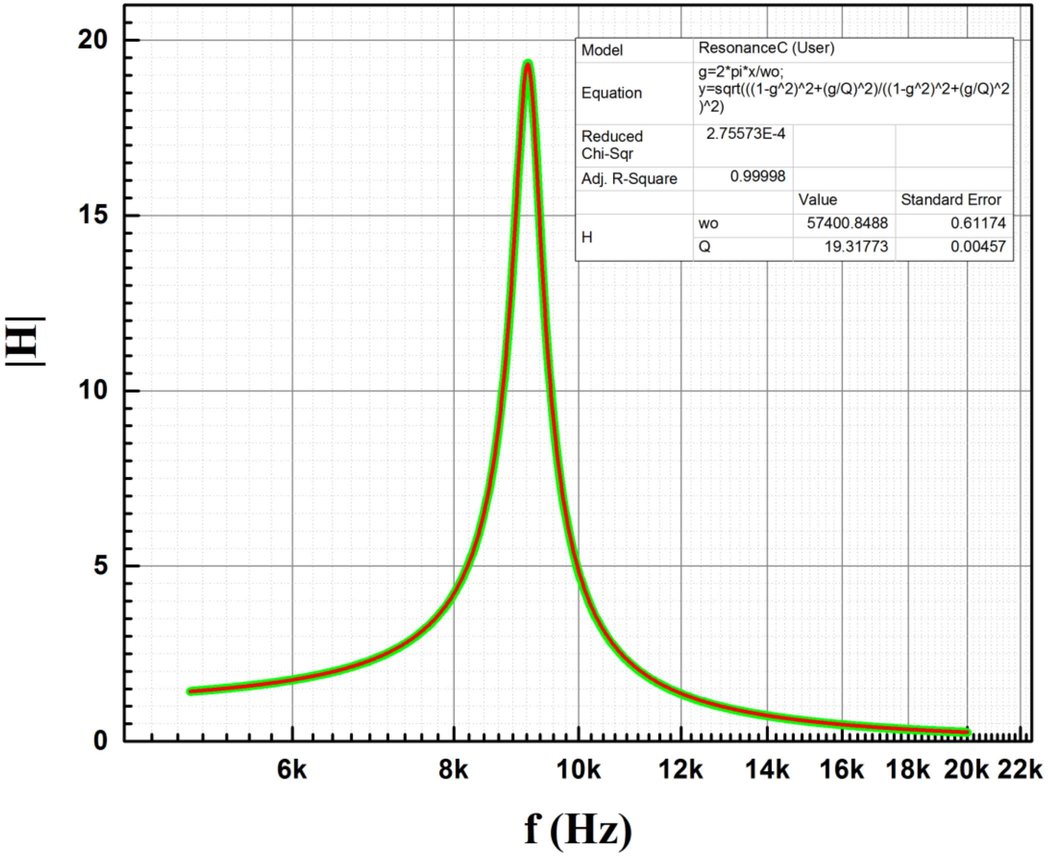
The resonance curves obtained on RLC circuits with different damping resistors.

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit



The resonance curves obtained on RLC circuits with different damping resistors

Fitting. RLC Resonance Circuit.



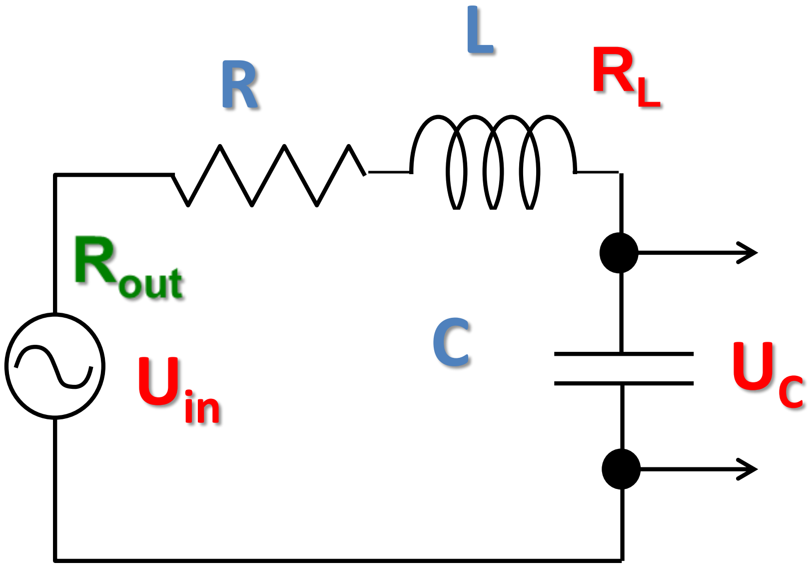
Model	ResonanceC (User)		
Equation	g=2*pi*x/wo; y=sqrt(((1-g^2)^2+(g/Q)^2)/((1-g^2)^2+(g/Q)^2)^2)		
Reduced Chi-Sqr	2.75573E-4		
Adj. R-Square	0.99998		
		Value	Standard Error
H	wo	57400.8488	0.61174
	Q	19.31773	0.00457

$$|H| = \frac{\sqrt{(1-\gamma^2)^2 + \left(\frac{\gamma}{Q}\right)^2}}{\left(1-\gamma^2\right)^2 + \left(\frac{\gamma}{Q}\right)^2}; \gamma = \frac{\omega}{\omega_0}$$

fitting function for |H|

variable parameters:
ω_0 and Q

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit

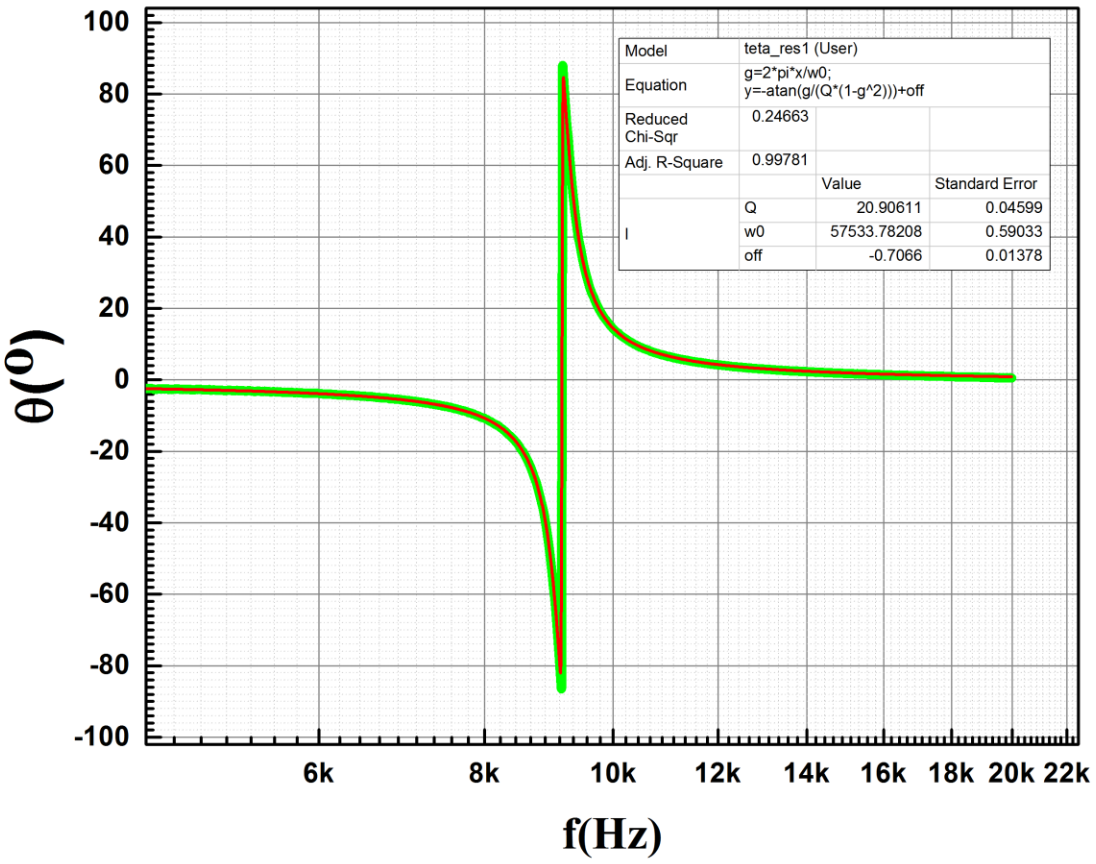


Actual damping resistance is a sum of R, R_L (resistance of the coil) and R_{out} (output resistance of the function generator)

$R=0$; $R_L = 35.8\Omega$; $R_{out}=50\Omega$

Actual R calculated from fitting pars is~88.8Ω what is reasonable close to 85.8Ω

Fitting. RLC Resonance Circuit.



Model	teta_res1 (User)		
Equation	g=2*pi*x/w0; y=-atan(g/(Q*(1-g^2)))+off		
Reduced Chi-Sqr	0.24663		
Adj. R-Square	0.99781		
		Value	Standard Error
I	Q	20.90611	0.04599
	w0	57533.78208	0.59033
	off	-0.7066	0.01378

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

measured

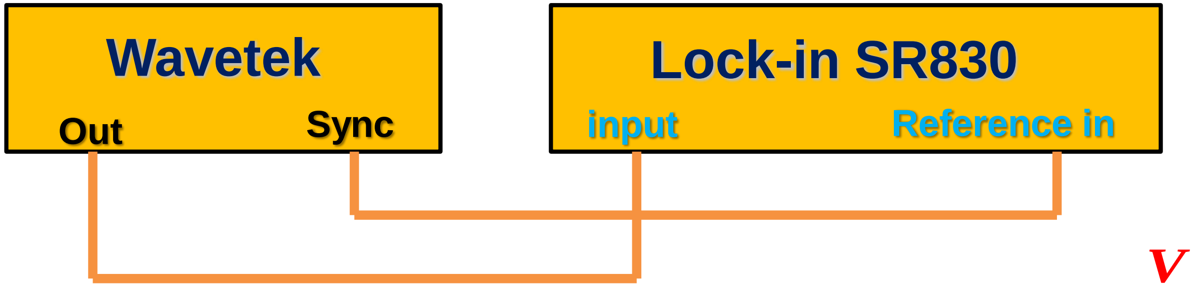
$$\theta = -\tan^{-1}\left(\frac{\gamma}{Q\left(1-\gamma^{2}\right)}\right);\gamma = \frac{\omega}{\omega_{0}}$$

fitting function

variable parameters:

ω_0 and Q

From Time Domain to Frequency Domain. Experiment.



Wavetek

Out Sync

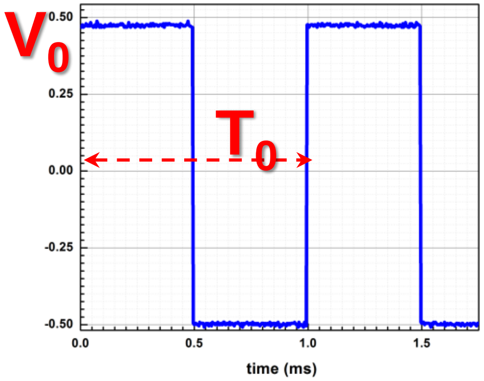
Lock-in SR830

input Reference in

F(t) – periodic function $F(t)=F(t+T_0)$:

$$V = V_0 \left(0 < t \le \frac{T_0}{2}\right);$$

$$-V_0 \left(\frac{T_0}{2} < t \le T_0\right)$$



Time domain pattern

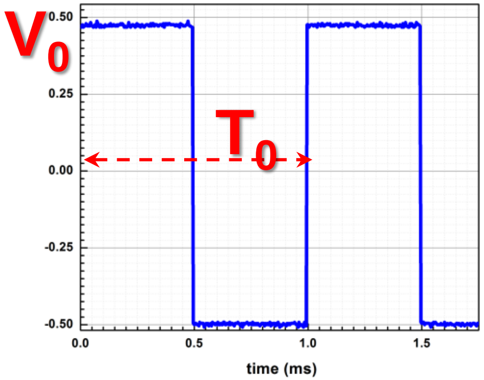
Frequency domain ?

$$a_n = \frac{2}{T_0}\int_0^{T_0} F(t)\cos\left(\frac{2\pi n t}{T_0}\right)dt;$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} F(t)\sin\left(\frac{2\pi n t}{T_0}\right)dt;$$

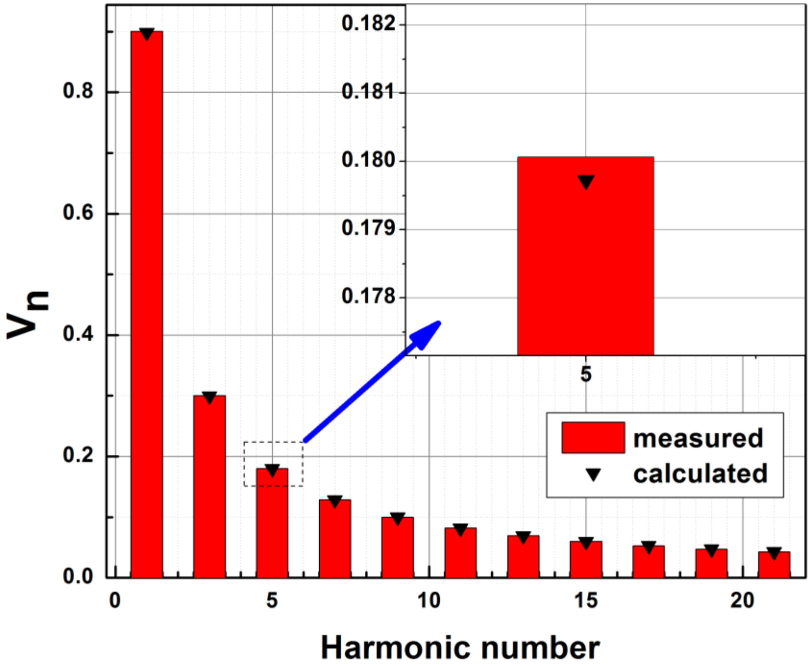
$$a_0 = \frac{2}{T_0}\int_0^{T_0} F(t)dt$$

From Time Domain to Frequency Domain. Experiment with SR830. Results.



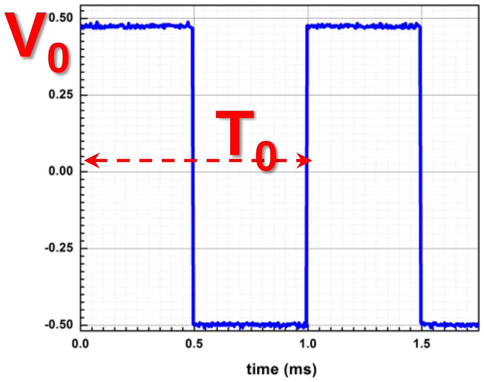
Time domain

Spectrum measured by SR 830 lock-in amplifier



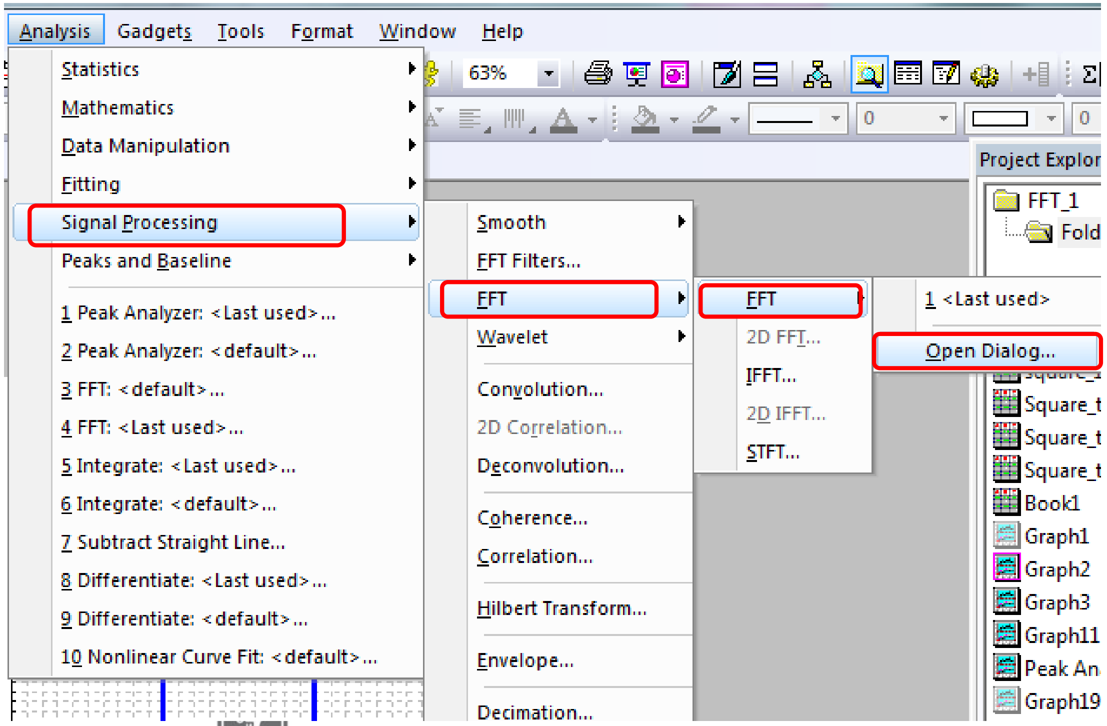
Frequency domain

From Time Domain To Frequency Domain. FFT using Origin. Results.

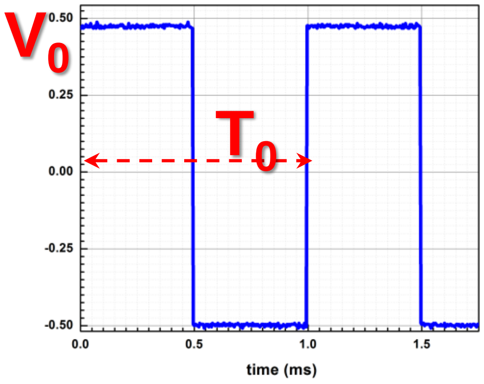


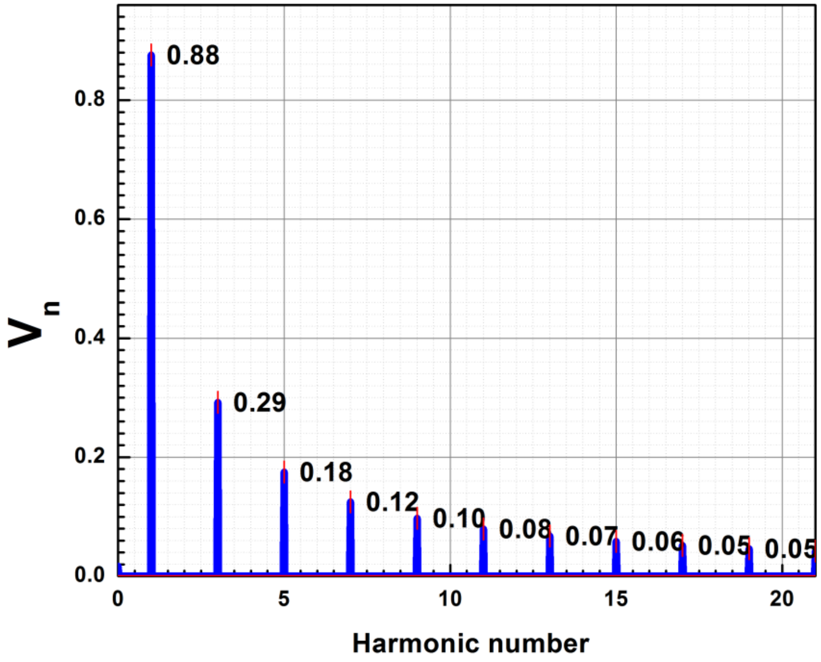
Time domain taken by Tektronix scope

Data file can be used to convert time domain to frequency domain



From Time Domain to Frequency Domain. FFT using Origin. Results.

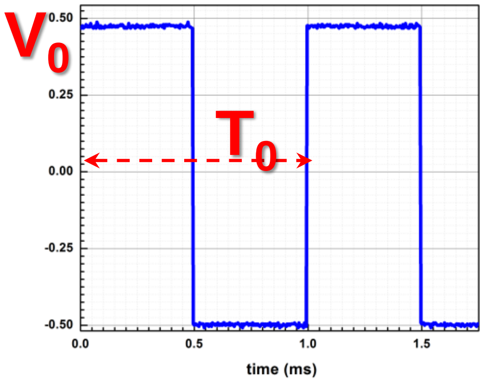




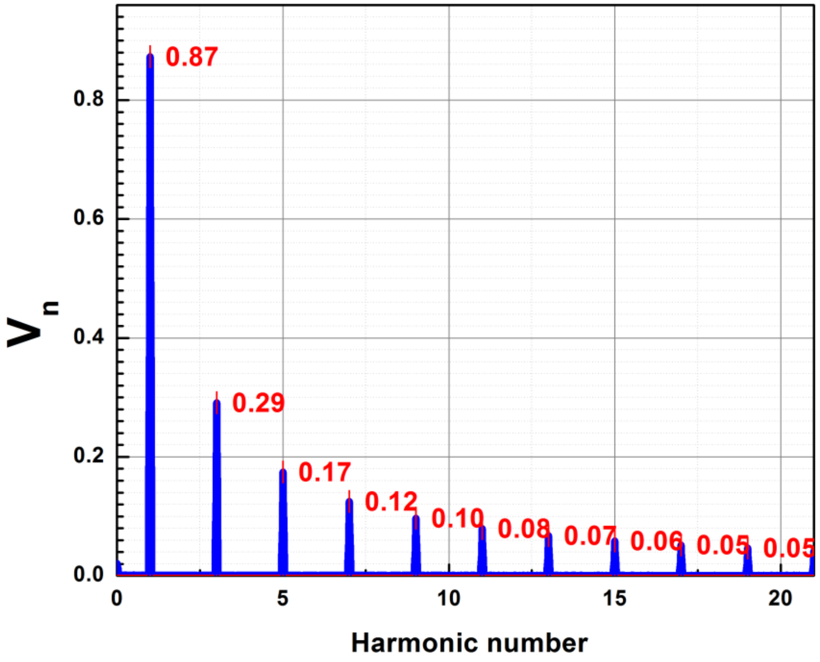
Time domain taken by Tektronix scope

Spectrum calculated by Origin. Accuracy is limited because of the limited resolution of the scope

From Time Domain to Frequency Domain. Using of the Math Option of the Scope.

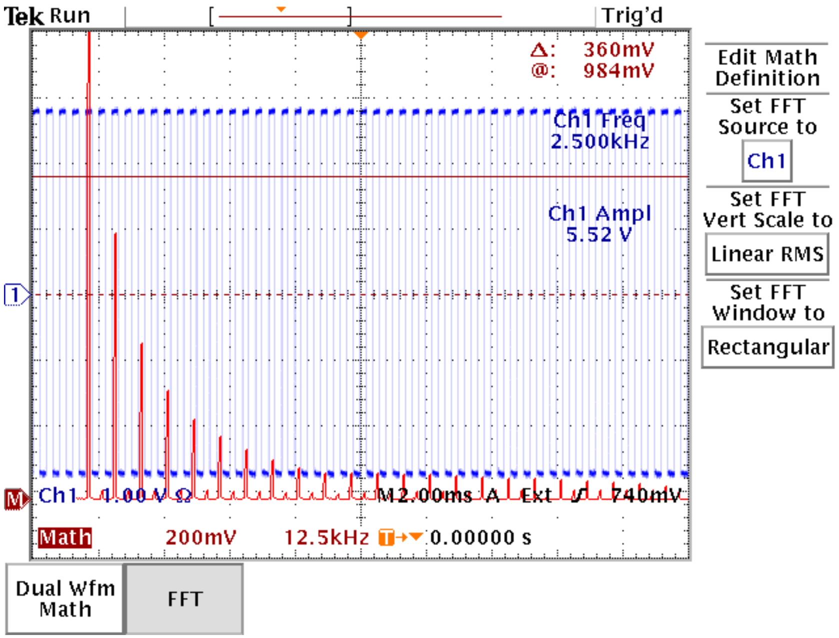


Time domain taken by Tektronix scope



Spectrum calculated by Tektronix scope.
Accuracy is limited because of the limited resolution of the scope

From Time Domain to Frequency Domain. Using of the Math Option of the Scope.



Δ: 360mV
@: 984mV

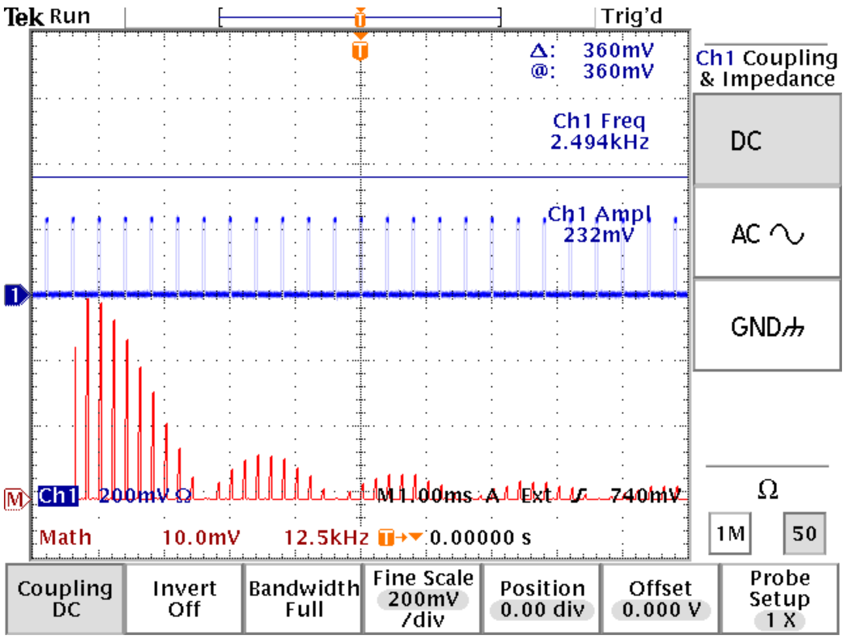
Ch1 Freq
2.500kHz

Ch1 Ampl
5.52 V

Ch1 1.00 V Ω M 2.00ms A Ext ⨍ 740mV

Math 200mV 12.5kHz 0.00000 s

Spectrum of the square wave signal



Δ: 360mV
@: 360mV

Ch1 Freq
2.494kHz

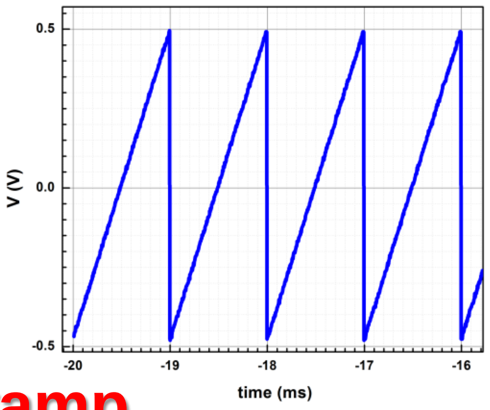
Ch1 Ampl
232mV

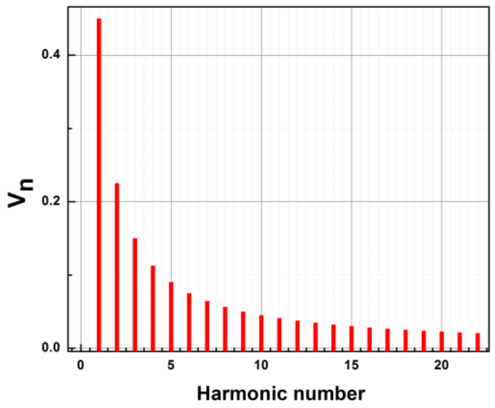
Ch1 200mV Ω M 1.00ms A Ext ⨍ 740mV

Math 10.0mV 12.5kHz 0.00000 s

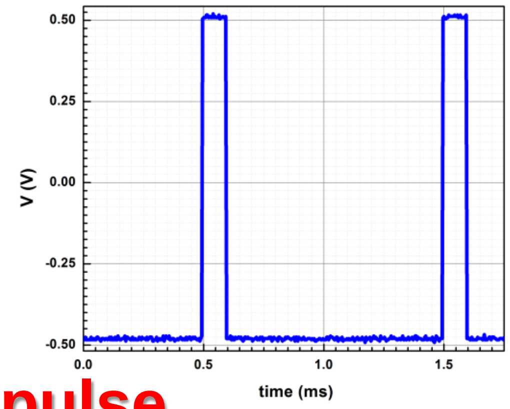
Spectrum of the pulse signal

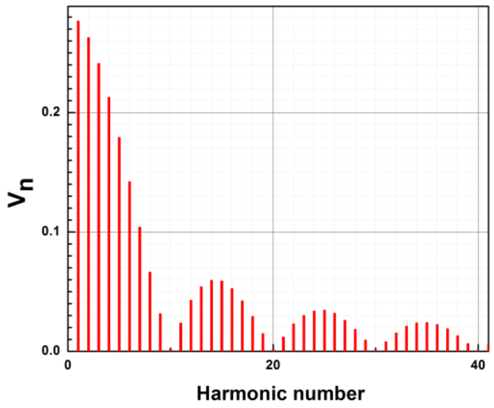
From Time Domain to Frequency Domain. Different Waveforms. Using Lock-in.





ramp

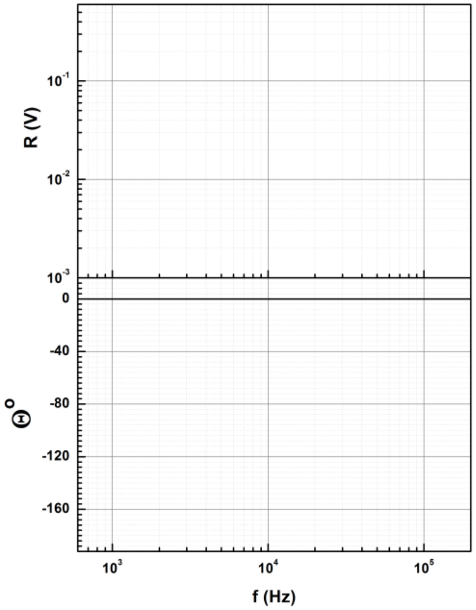


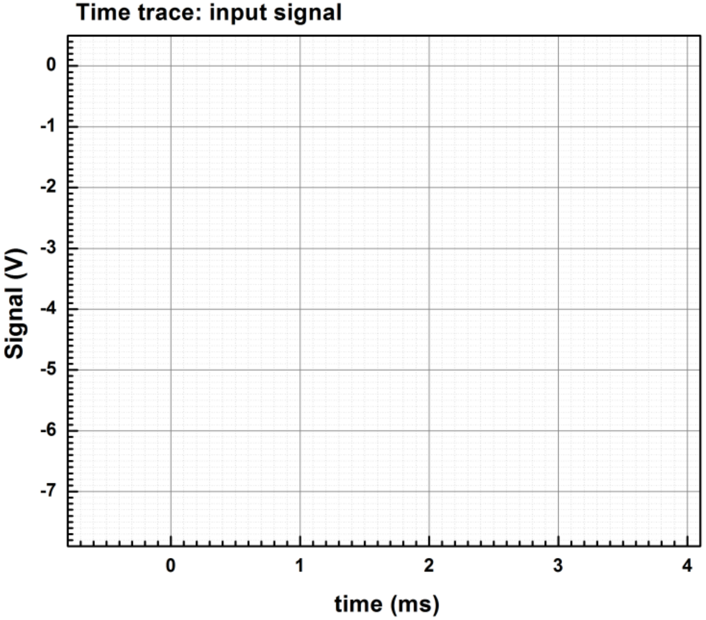


pulse

Appendix #1

Origin templates for the this week Lab:




Time trace: input signal

Physics401

References:

1. **John H. Scofield, “A Frequency-Domain Description of a Lock-in Amplifier” *American Journal of Physics* 62 (2) 129-133 (Feb. 1994).**
2. ***Steve Smith* “The Scientist and Engineer's Guide to Digital Signal Processing*” copyright ©1997-1998 by Steven W. Smith. For more information visit the book's website at: www.DSPguide.com" ****

- **You can find a soft copy of this book in:**
- *\\engr-file-03\PHYINST\APL Courses\PHYCS401\Experiments\DSP and FFT*

Appendix. Using OriginPro for fitting

Some recommendations how to use OriginPro nonlinear fitting option

You can find some examples of OriginPro projects and some recommendation how to do the analysis in next folder:

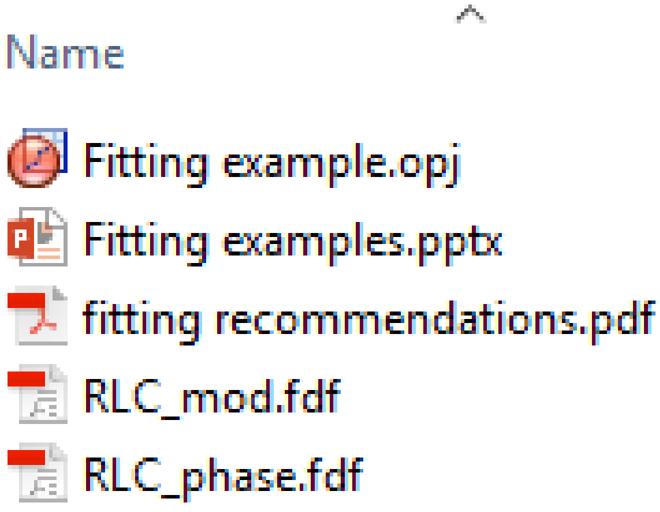
\\engr-file-03\PHYINST\APL Courses\PHYCS401\Students\3. Frequency Domain Experiment. Fitting

An example of Origin project

Slides with results of fitting for all studied in this Lab RLC circuits

Some recommendations how to write the fitting function and start the fitting process

Examples of fitting functions which can be used for fitting in Origin



Name

Fitting example.opj

Fitting examples.pptx

fitting recommendations.pdf

RLC_mod.fdf

RLC_phase.fdf